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Mirzakhani's recursion relations and the Witten conjecture

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1 Introduction

A remarkable feature of many topological field theories is their intrinsic connection to geometry. Famous examples are Donaldson theory (e.g. [10]), where one connects a SU(2)-gauge field theory to invariants of 4-manifolds; Chern–Simons theory (e.g. [35]), where a gauge field theory with Chern–Simons action is connected to knot invariants; and topological string theory (e.g. [21]), where a string theory is connected to the calculation of Gromov–Witten invariants of J-holomorphic curves. This correspondence between quantum field theories and similar approaches to physics and geometry is one of the most intimate connections between mathematics and physics. It is the purpose of this Master thesis to describe one such correspondence in great detail at least from the mathematical side.

In fact we will be dealing with a problem related to topological string theory. As is known, string theory is one approach to unify quantum field theory and gravity. It is the gravitational sector that we will be interested in later. In fact, we will talk about expectation values of certain purely gravitational states and see that we can actually compute them with a recursion relation for all genera. Furhtermore we will see that the generating function for these expectation values will satisfy certain relations which are well-known from another theory of quantum gravity, namely matrix models. In general, the transition between physical and geometrical picture is done by BRST-quantization in which one identifies certain geometric differential operators with BRST operators. Very explicit references are e.g. [2] and [34].

On the mathematical side we will investigate a certain symplectic structure on the moduli space of Riemann surfaces. We will use a standard technique from toric geometry to calculate intersection numbers of Chern classes of torus bundles via the Duistermaat–Heckmann theorem. The torus actions will be given by rotation of a marked point on a curve on a hyperbolic surface representing a point in moduli space. The symplectic form will be in terms of the local hyperbolic structure and the momentum map will be the hyperbolic length function. So technically we will be talking about hyperbolic surfaces most of the time, create suitable actions and then calculate the volume of the moduli space with respect to the symplectic form, knowing that the coefficients give the desired intersection numbers.

Although it will remain unclear until Section 5.2 what the exact correspondence between the physical picture (i.e. topological gravity as the gravitational sector of topological field theory) and the geometric one (i.e. intersection theory on the Deligne–Mumford moduli space) is, we will put much effort in defining and proving the necessary statements for understanding the geometric idea. Before explaining how the thesis is organized I want to give a small historical and bibliographical introduction into the topic.

The Witten conjecture is a conjecture for the generating function of ψ-intersection numbers on the Deligne–Mumford moduli space. Edward Witten proposed it in 1990 in [], because he noted that the geometrical theory of intersection numbers can be seen as a two dimensional quantum gravity theory. But since there was already a description of two dimensional quantum gravity in terms of matrix models he conjectured that the two would be two descriptions of the same physical theory, i.e. their correlation functions should coincide. This led him to conjecture that the generating function F of ψ-intersection numbers satisfies a set of relations, namely $L_k F = 0$ for L_k some differential operator with $k \geq -1$. These operators were known to satisfy half a Virasoro algebra which is again equivalent to F being a solution to the KdV-equation. One can see that this conjecture linked many different areas of physics with mathematics and was thus very inspiring even for mathematiciens. The conjecture was proven by Maxim Kontsevich in 1991 in [20] by a brute-force calculation using a lot of combinatorics in matrix models. Since then there have been many new proofs using different techniques from algebraic geometry (see e.g. [19]) and differential geometry (see e.g. [28] and [25]). It is this last point that we want to explain further,

i.e. Mirzakhani's proof of the Witten conjecture in 2001. It uses techniques from symplectic geometry and is very beautiful. Remembering that this conjecture applies to the gravitational sector of topological string theory one could ask whether it generalizes to the case of matter fields, i.e. intersection numbers of certain classes on the moduli space of J-holomorphic curves. In terms of this description the corresponding question (the so-called Virasoro conjecture) is whether the generating function for Gromov–Witten invariants F satisfies the Virasoro condition $L_k F =$ with $k \geq -1$ for certain differential operators. This would allow one to calculate many Gromov–Witten invariants much easier, similarly to the recursive calculation of the intersection numbers of the ψ-classes. In fancy language one could say that in this thesis we prove the Virasoro conjecture for a point, meaning that we forget about the J-holomorphic curves because we set them all constant. Except for special cases this more general conjecture is unproven until today.
In this thesis we will mainly follow Mirzakhani's proof from [25] and [24]. However, many gaps will be filled in and some objects will be introduced in more detail such that any reader with basic geometric knowledge should be able to follow the exposition.
In Section 2 we will prove the following McShane identity for lengths of simple closed geodesics on hyperbolic surfaces. Although this identity may seem totally unrelated to intersection numbers of physics it will descend to an equation on the moduli space of Riemann surfaces and can be used further there. This section includes a description of the most important tools from hyperbolic geometry. Thus the main result of this section is

Theorem 1 (McShane identity)**.** *For a hyperbolic surface X with geodesic boundary $\beta_1, \dots, \beta_n$ of legnths $L_1, \dots, L_n$ one has*

$$\sum_{\{\gamma_1,\gamma_2\}\in\mathcal{F}_1} \mathcal{D}(L_1, l_{\gamma_1}(X), l_{\gamma_2}(X)) + \sum_{i=2}^{n} \sum_{\gamma\in\mathcal{F}_{1,i}} \mathcal{R}(L_1, L_i, l_\gamma(X)) = L_1,$$

where $\mathcal{D}$ and $\mathcal{R}$ are explicitely known functions and $\mathcal{F}_1$ denotes the set of simple closed non-homotopic pairs of curves bounding an embedded pair of pants with β_1 and $\mathcal{F}_{1,i}$ the set of all simple closed curves bounding an embedded pair of pants with β_1 and β_i. Furthermore $l_\gamma(X)$ denotes the hyperbolic length of the unique geodesic representative in the free homotopy class of γ.

Section 3 deals with Teichmüller theory. Here we will define the Teichmüller and moduli spaces as well as Fenchel–Nielsen coordinates and the Weil–Petersson metric. These last two objects will be very important for the proceeding. The main result of this section, which we will not prove, is

Theorem 2 (Fenchel–Nielsen coordinates and Wolpert's theorem)**.** *For the Teichmüller space of bordered Riemann surfaces $\mathcal{T}_{g,n}(L)$ and the Weil–Petersson symplectic form on this space one has*

$$\begin{aligned} \mathcal{T}_{g,n}(L) &\simeq \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n} \ni (l,\tau) \\ \omega_{WP} &= \sum_{i=1}^{3g-3+n} dl_i \wedge d\tau_i \end{aligned}$$

The next section is the main section of this work. There we will use the McShane identity on the moduli space and introduce a technique in order to integrate functions of the kind of the McShane identity over the moduli space. In fact we will introduce a certain cover of the moduli space and pull back the integration there. At the end we deduce a recursion relation for the Weil–Petersson volume of the moduli space. Then we will see that it is actually a polynomial in the lengths of the boundary curves. We will see a couple of examples for the calculation of the coefficients which will be linked to the intersection numbers in the fifth section. The main result is the complete determination of the Weil–Petersson volumes of the moduli space given by Mirzakhani's recursion relation.

Theorem 3 (Mirzakhani's recursion relation)**.** *For the moduli space of bordered Riemann surfaces $\mathcal{M}_{g,n}(L)$ with geodesic boundary of length $L_1, \dots, L_n$ and its Weil–Petersson volume $V_{g,n}(L)$ for*

$(g,n) \neq (0,3),(1,1)$ we have

$$\frac{\partial}{\partial L_1} L_1 V_{g,n}(L) =$$
$$\frac{1}{2}\int_0^\infty \int_0^\infty xy \left(\frac{V_{g-1,n+1}(x,y,L_2,\dots,L_n) H(x+y,L_1)}{2^{m(g-1,n+1)}} \right.$$
$$\left. + \sum_{a\in\mathcal{I}_{g,n}} \frac{V_{g_1,n_1+1}(x,L_{I_1})}{2^{m(g_1,n_1+1)}} \frac{V_{g_2,n_2+1}(y,L_{I_2})}{2^{m(g_2,n_2+1)}} H(x+y,L_1) \right) dx\ dy$$
$$+ \frac{2^{-m(g,n-1)}}{2} \sum_{j=2}^{n} \int_0^\infty x\left(H(x,L_1+L_j) + H(x,L_1-L_j)\right) V_{g,n-1}(x,L_2,\dots,\widehat{L_j},\dots,L_n)dx,$$

where the sum over $a \in \mathcal{I}_{g,n}$ ranges over possible configurations $((g_1,I_1),(g_2,I_2))$ of cut surfaces obtained by removing an embedded pair of pants containing β_1 and H is an explicitely known kernel function related to $\mathcal{D}$ and $\mathcal{R}$. Together with $V_{1,1}(L) = \frac{\pi^2}{6} + \frac{L^2}{24}$ and $V_{0,3}(L) = 1$ this determines all Weil–Petersson volumes.

In Section 5 we will compactify the moduli space of Riemann surfaces to the Deligne–Mumford orbifold and introduce certain line bundles over this space. Then we introduce a space on which the moduli spaces arise as symplectic quotients such that we can use Duistermaat–Heckmann to deduce a relation between the volumes of the symplectic quotients and the integrals of Chern classes of some torus bundles. However, having already calculated the Weil–Petersson volume we can compare the coefficients in front of the lengths of the boundary components and see that we in fact calculated the intersection numbers of the corresponding cohomology classes. We then relate this via the BRST formalism to physical correlation functions. Despite the correspondence to physics the main result of this section is

Theorem 4 (Connection to intersection numbers)**.** *For the Chern classes of the tautological line bundles $\mathcal{L}_i$ for $i = 1,\dots,n$ over $\overline{\mathcal{M}}_{g,n}$, i.e. $\psi_i := c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ we have*

$$V_{g,n}(L) = \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\leq 3g-3+n}} \frac{2^{m(g,n)|\alpha|}}{2^{|\alpha|}\alpha!(3g-3+n-|\alpha|)!} \int_{\overline{\mathcal{M}}_{g,n}} \psi^\alpha \omega_{WP}^{3g-3+n-|\alpha|} L^{2\alpha},$$

where α is a mutli-index and $m(g,n)$ is 1 if $(g,n) = (1,1)$ and 0 otherwise.

To end this introduction I want to mention that there are some details which will not be proven and which in fact have turned out to be technically more difficult than expected. The first such thing is the analytical construction of Teichmüller and moduli spaces. Since we need here moduli spaces of surfaces with boundary this requires modification of the usual theory, however, as it is already very elaborate for the case of closed surfaces of some genus we will not do any concrete calculations but refer to standard sources. Second, we will not introduce orbifolds although this is necessary because the moduli space is naturally an orbifold as the action of the mapping class group on Teichmüller space is not free. Thus we will try to make all the statements on suitable coverings which are manifolds and then give arguments why the results descend to the quotient. This was done in [25], too, in order to avoid technical issues. A more complete description would involve the description of orbifolds in the language of groupoids which would have been far too much to describe here. The third point is the construction of the tautological bundles in Section 5.1.2. It turned out that the construction of the actual bundle structure was not that easy even on Teichmüller space. Thus we will have to make assumptions or basically accept that those bundles exist. The fourth point is a certain limit procedure which seems intuitively clear but which causes technical problems. Using Duistermaat–Heckmann we will relate the Weil–Petersson volumes for moduli spaces of boundary lengths L to those with punctures. However, the last ones are of a different kind although they look like the limit $L \longrightarrow 0$. We will assume that this limit exists and actually gives the right objects.

Apart from these problems the proof should be concise and complete.

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2 The McShane identity

2.1 Background

2.1.1 Hyperbolic geometry

Hyperbolic geometry refers to the investigation of hyperbolic structures on differentiable manifolds. The term hyperbolic means that in some sense the manifold locally looks like a hyperbola. One interesting feature of hyperbolas is that they have a sectional curvature constantly equal to minus one. This property implies lots of corollaries, some of which we have collected in this section. However, the main point of this work is to prove an identity which holds on all surfaces which admit a Riemannian metric with sectional curvature of minus one. Since this is a wide topic we will not prove most of the statements but only cite them such that we can work with them.

Definition 2.1. Let M be an orientable differentiable manifold. A hyperbolic structure on M is a Riemannian metric g on M with constant sectional curvature minus one.

We will deal only with surfaces, i.e. manifolds of dimension two. Using a standard representation of hyperbolic space (i.e. for example $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ with $g_{\text{hyp}} = \frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$ for $z = x + \mathrm{i}y$) it can be shown that this definition is equivalent to the following one:

Definition 2.2. A hyperbolic surface M is a topological surface with a (maximal) atlas of coordinate charts such that for each transition map of coordinate charts $\phi_i : U_i \longrightarrow V_i \subset \mathbb{H}^2$ $\phi_i \circ \phi_j^{-1} : V_i \cap V_j \longrightarrow V_i \cap V_j$ restricts to an orientation preserving isometry of $(\mathbb{H}^2, g_{\text{hyp}})$.

Of course the standard hyperbolic space can also be chosen to be the Poincaré model or the hyperboloid model, but we will mainly do calculations on $\mathbb{H}^2$. One important consequence of this definition is that the universal cover of a hyperbolic surface can be chosen to be $\mathbb{H}^2$ and one may require the projection $\pi : \mathbb{H}^2 \longrightarrow M$ to be an isometry in order to obtain a hyperbolic metric on M. Thus we may lift to $\mathbb{H}^2$ in order to simplify calculations.
For the converse direction we require M to be compact such that it has finite area with respect to its hyperbolic metric. In this case it is possible to find a discrete subgroup $\Gamma \subset \operatorname{Isom}^+(\mathbb{H}^2)$, where $\operatorname{Isom}^+(\mathbb{H}^2)$ are the orientation preserving isometries of $\mathbb{H}^2$, such that M is isometric to $\mathbb{H}^2 \diagup \Gamma$.
Thus we conclude

Theorem 2.3. *Let M be a compact hyperbolic surface. Then $M \simeq \mathbb{H}^2 \diagup \Gamma$, where $\Gamma \subset Isom^+(\mathbb{H}^2)$.*

Proof. See [4]. $\square$

Furthermore we see that $\Gamma \simeq \pi_1(\mathbb{H}^2 \diagup \Gamma) \simeq \pi_1(M)$ because $\mathbb{H}^2$ is simply connected.

Theorem 2.4. *On a hyperbolic surface M there is a bijection between closed geodesics and free homotopy classes of curves, i.e. in each free homotopy class $[\alpha] \in \pi_{free}(M)$ there is a unique geodesic representative.*

Proof. See [4]. $\square$

Remark 2.5. This is of course very different to flat or spherical structures. However, the element $[\alpha]^2$ corresponds to the same geodesic run through twice which means that not all geodesics are simple. Later we will be mainly interested in simple closed geodesics. The geodesic representative of a free homotopy class of a simple curve is again simple, see again [4].

Since we will do many calculations in the universal cover of M we should say a couple of things about lifts. In principle any space isometric to $\mathbb{H}^2$ can be chosen as a lift, thus in particuluar we can use an isometry of $\mathbb{H}^2$ in order to obtain a lift with special representatives of certain geodesics. For example if we wanted to do calculations close to a certain closed geodesic on M we could choose the lift in such a way that this geodesic lifts to a vertical line (which is a geodesic) in $\mathbb{H}^2$. This choice of the lift may simplify many calculations.
One may ask which Riemann surface admits a hyperbolic structure. Since diffeomorphsim classes of compact Riemann surfaces are determined by the genus and the number of boundaries (if any) one can use the index theorem of Gauss–Bonnet to determine the condition. We have

Theorem 2.6 (Gauss–Bonnet)**.** *Let M be a Riemann surface with boundary equipped with a Riemannian metric g. Denote by K its sectional curvature, by k_g the geodesic curvature of a curve, by dA its volume form and by ds the induced volume form on some curve. Then*

$$\int_M K\,dA + \int_{\partial M} k_g\,ds = 2\pi\chi,$$

where χ is the Euler characterisitics of M, i.e. $\chi = 2 - 2g + n$, where g denotes the genus, and n the number of boundaries.

Proof. See [7]. □

Remark 2.7. If the boundary is geodesic we have $k_g = 0$ and for hyperbolic metrics $K = -1$ implies $-2\pi\chi = \operatorname{Area}(M) > 0$ which means that a necessary condition is that the surface has a negative Euler characteristics. This is indeed sufficient, as the next lemma states.

Lemma 2.8. *A compact Riemann surface with genus g and n boundaries admits a hyperbolic structure if and only if $2g - 2 + n > 0$.*

Proof. See [18]. □

Definition 2.9. A hyperbolic surface with boundary is a punctured Riemann surface with boundary together with a hyperbolic metric such that the boundaries are geodesics with respect to this metric and the punctures correspond to cusps of the hyperbolic metric. We will require the surfaces to have finite area.

Remark 2.10. Although the cusps can be seen intuitively as limits of boundaries whose length becomes zero we have to take care because the two cases need in principal different treatments. However, for our purposes it does not cause any problems, because one can do the following trick: Replace the cusp by a horocycle close to it and remove the cusp. If it is small enough the geodesics will all point perpendicular to it into the direction of the cusp and all statements about geodesics going up the cusp may be replaced by statements on geodesics meeting the boundary (i.e. the horocycle) perpendicular. In this way one has to modify all the statements in the following, however, we will not do this explicitly. Instead the final statement for surfaces with punctures can be obtained by considering the limit of the lengths going to zero and looking at its first derivative, see [25].

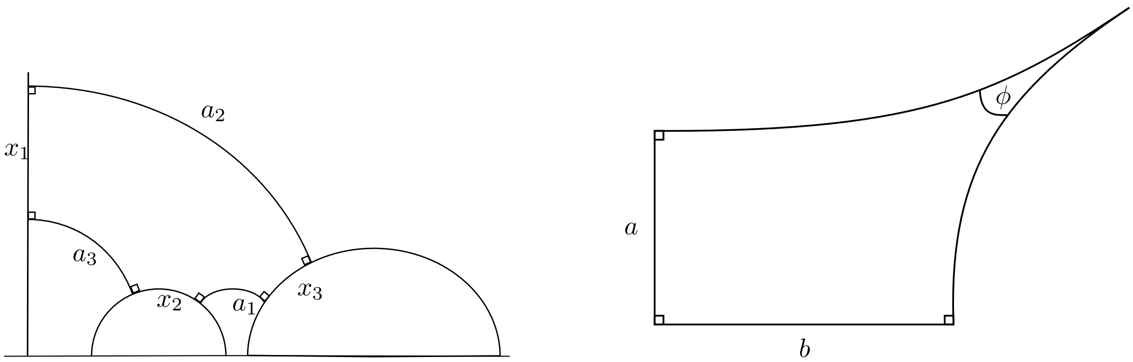
2.1.2 Hyperbolic plane

As was already mentioned we will do many calculations in the hyperbolic plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ with the metric $g_{\text{hyp}} = y^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2)$ for coordinates $z = x + \mathrm{i}y$. This is a hyperbolic metric because its sectional curvature is minus one. As is well known the geodesics are given by the vertical lines as well as half-circles whose center lies on the real axis, see [4]. Since the metric is proportional to the Euclidean metric by a positive factor, angles measured in both metrics are the same.
Since we will be dealing with hyperbolic pairs of pants we need the following statement about hexagons.

Lemma 2.11. *Pick three positive numbers x_1, x_2 and x_3. Then there is a unique hyperbolic hexagon (up to isometry) with geodesic boundaries of alternating lengths x_1, x_2 and x_3. The remaining sides are determined by*

$$\cosh a_3 = \frac{\cosh x_3 + \cosh x_1 \cosh x_2}{\sinh x_1 \sinh x_2}$$

and cyclic permutations, where a_3 is the side as in Figure 2.1a.



(a) A hyperbolic hexagon in the upper half plane

(b) A hyperbolic (tri)rectangle

Figure 2.1

Proof. See [4]. ☐

Lemma 2.12. *For a rectangle in the hyperbolic plane with sides a and b opposite to the angle ϕ and where all other angles are rectangular, see Figure 2.1b, one has*

$$\cos(\phi) = \sinh a \sinh b.$$

This remains true if ϕ becomes an ideal angle, i.e. $\phi = 0$.

Proof. See [4]. ☐

2.1.3 Laminations

We will now introduce objects which will become very important later on, so-called laminations. In fact we are interested in simple complete geodesics on hyperbolic surfaces which are related to laminations. Here, simple means that the geodesic does not intersect itself, i.e. the map $\gamma : \mathbb{R} \longrightarrow M$ is injective. Complete means that on a surface without boundary it is defined for all times, i.e. on whole $\mathbb{R}$. If we have a boundary then we also consider geodesics perpendicular to boundaries as complete (as they would be complete on the doubled surface).

It is possible to distinguish simple complete geodesics on hyperbolic manifolds by the behaviour of their ends. Such an end may spiral into a set called a lamination. Such a lamination consists of a union of geodesics. Since the behaviour of the geodesic spiraling into the lamination is connected to the lamination we will have to talk about these. The material collected here is mainly based on [6] and [5].

Definition 2.13. A lamination Λ on a hyperbolic surface M is a closed subset $\Lambda \subset M$ which is the disjoint union of complete simple geodesics on M, i.e.

$$\Lambda = \dot{\bigcup_{\gamma}} \mathrm{Im}(\gamma).$$

The geodesics as subsets of Λ are called leaves of the lamination.

Example 2.14. The easiest example of a lamination is a finite union of disjoint closed simple geodesics, see Figure 2.2. Including infinite simple geodesics which spiral into those closed geodesics we obtain slightly more complicated examples.

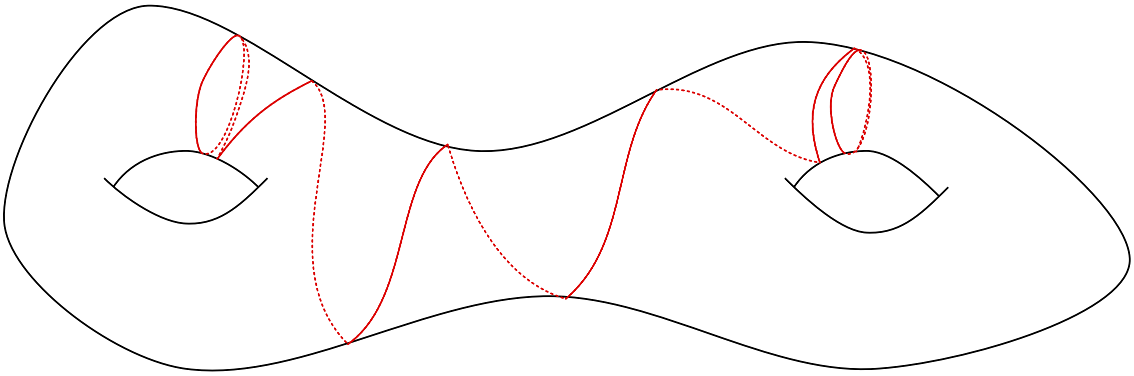


Figure 2.2: An example of a lamination consisting of two closed simple geodesics and one infinite geodesic spiraling into those two geodesics. Examples of laminations which are not closed simple geodesics union geodesics that spiral into them are hard to draw. However, they exist, see [6], as they are limits of sequences of simple closed geodesics of increasing length

However, the most interesting examples are of a different type. Before examining the general structure of laminations we will recall some basic facts about laminations.

Lemma 2.15. *The closure of a disjoint union of simple geodesics is a lamination.*

Proof. This can be shown by picking a sequence of points in the union together with the direction of their corresponding geodesics. Those directions give a sequence in $\mathbb{R}P^1$ which is compact and thus contains a convergent subsequence. One then shows that the geodesic defined by this limit point lies in the closure and is a simple geodesic not intersecting any other leaf. See [6] for more details. □

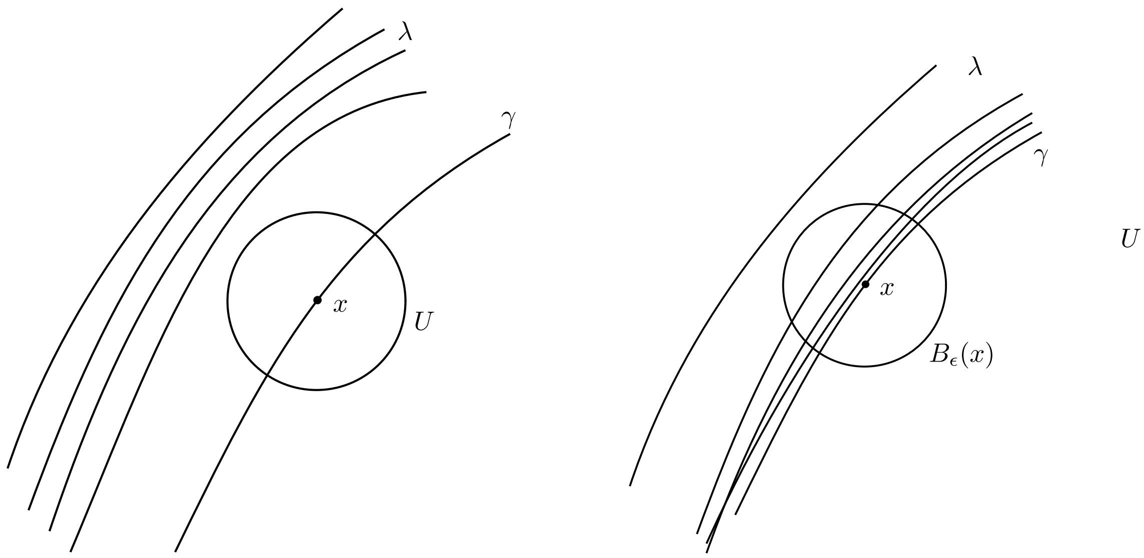
Considering the set of closed subsets $C(M)$ of the manifold M with the Hausdorff distance as a metric one can investigate the subset $\mathrm{Lam}(M)$ which consists of all geodesic laminations of M, see [5]. In [5] it is shown that $\mathrm{Lam}(M) \subset C(M)$ is compact with respect to the Hausdorff metric. Recalling that for any $L > 0$ there are only finitely many closed simple geodesics of length less than L we see that there exist laminations of a different kind as one can pick a sequence of closed simple geodesics with increasing length. Such a sequence converges in $\mathrm{Lam}(M)$ with respect to the Hausdorff metric to something which is not a closed simple geodesic.
One important question is whether a lamination can be dense somewhere on M. For example on the flat torus it is well known that a simple closed geodesic that wraps around the torus can be moved continuously in the transversal direction and remains a closed simple geodesic. However, since such a curve is homotopic to the first curve it could not be another geodesic on a hyperbolic surface by Theorem 2.4. In general, the following statement holds.

Lemma 2.16. *Laminations of closed oriented hyperbolic surfaces are nowhere dense, i.e. their interior is empty.*

Proof. See [6]. □

In order to continue this list of properties of geodesic laminations we need some more definitions as follows.

Definition 2.17. Let M be a hyperbolic surface and $\Lambda \subset M$ a geodesic lamination. Then



(a) Isolated leaf

(b) Boundary leaf

Figure 2.3: These images illustrate the local picture one has in mind for boundary and isolated leaves

1. a leaf $\gamma \subset \Lambda$ is called isolated if for all $x \in \gamma$ there exists $U \subset M$ such that $x \in U$, U is homeomorphic to a disc and $U \cap \gamma$ is homeomorphic to a diameter of this disc.

2. a leaf $\gamma \subset \Lambda$ is called a boundary leaf of a principal region U if U is a component of the complement $M \setminus \Lambda$ and if for all $x \in \gamma$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \cap U$ contains at least one component of $B_\epsilon(x) \setminus B_\epsilon(x) \cap \gamma$.

3. a nonempty Λ is called minimal if no proper subset of Λ is a geodesic lamination.

Here, $B_\epsilon(x)$ denotes the open ball of radius ϵ around x. These defintions are illustrated in Figure 2.3.
Now the following statements holds.

Theorem 2.18. *1. If $\Lambda \setminus \{\text{isolated leaves}\} = \emptyset$ then $\Lambda = \dot{\bigcup} \gamma$ where the disjoint union is over a finite set of disjoint complete simple geodesics.*

2. The set of boundary leaves of a lamination is dense in the lamination.

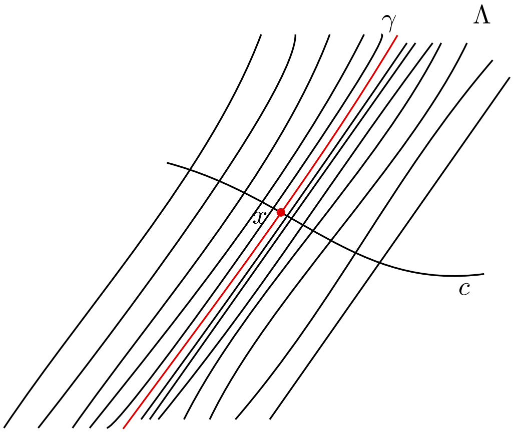
Proof. 1. See [5].

2. This can be easily seen by looking at the intersection of the lamination Λ with a transverse arc c as in Figure 2.4b. Arbitrarily close to the point $x \in \gamma \subset \Lambda$ there exists a point $u \in M \setminus \Lambda \cap c$ because Λ is nowhere dense. Then on c there exists a closest point in $\Lambda \cap c$ to u in the direction of x, call it v. It exists because the lamination is closed and thus if there was a sequence of points in $c \cap \Lambda$ converging to u we would have $u \in \Lambda$. The geodesic going through v, $\gamma_v \subset \Lambda$ is a boundary leaf. See [5] for more details.

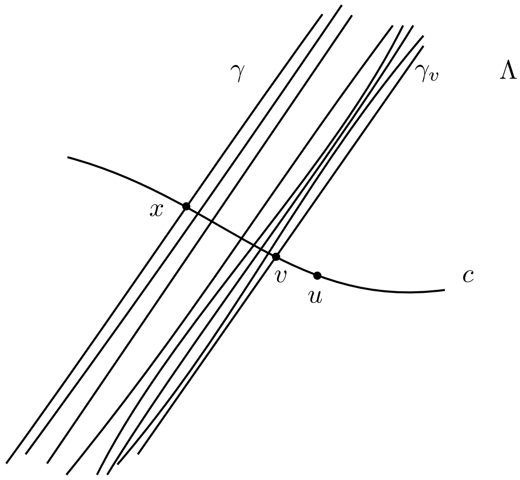
□

We will see soon that minimal laminations are in fact the building blocks of general laminations. Thus we are interested in those laminations.

Lemma 2.19. *If Λ is a minimal lamination then either Λ consists of a single geodesic (i.e. either it is closed or it is infinite with both ends in a cusp) or it consists of uncountably many leaves.*



(a) An arc c transverse to a lamination Λ



(b) A transverse arc to a boundary leaf of a lamination

Figure 2.4: The local picture of transverse arcs used in some proofs

Proof. Take a leaf $\gamma \subset \Lambda$. If it is isolated then $\Lambda \setminus \gamma = \emptyset$ and thus the first case applies. If it is not isolated we can consider a transvers arc c trough $x \in \gamma$, see Figure 2.4a. Since γ is not isolated, x is an accumulation point of $c \cap \Lambda$. Thus $c \cap \Lambda$ is closed and every point is an accumulation point which means that it is a perfect set and thus uncountable. However, every leaf of Λ intersects c only countably often and thus Λ must have uncountably many leaves. □

Theorem 2.20. *Let Λ be a geodesic lamination on the hyperbolic surface M. Then*

$$\Lambda = \dot{\bigcup_{\textit{finite}}} \{\textit{min. sublaminations of } \Lambda\} \cup \dot{\bigcup_{\textit{finite}}} \{\textit{isolated geodesics}\},$$

where the ends of the isolated geodesics each either spiral into a minimal lamination or go up a cusp.

So we know what an arbitrary geodesic lamination looks like and we can investigate the set $\mathrm{Lam}(M)$ a bit better.

Proof. In order to prove this structure theorem we cut the surface along a minimal lamination $\lambda \subset \Lambda$ and see that we obtain a possibly disconnected hyperbolic surface M' with a different lamination Λ'. Now the minimal sublaminations of L correspond to minimal sublaminations of Λ' where λ is replaced by some boundary leaves of M' which are minimal, see Figure 2.5. Continuing this cut procedure we obtain more boundary components but since the area of the surface is finite and we require the boundaries to be geodesic there is only a finite set of boundary components and thus of minimal sublaminations. So at the end all minimal sublaminations are contained in the boundary which can be shown to imply that there are only finitely many leaves of the remaining lamination. Thus also the set of isolated geodesics is finite, see [5] for more details. □

Theorem 2.21. *The set of finite laminations is dense in the set of all laminations, i.e. every lamination can be approximated by a finite one.*

Proof. See [5]. □

By the structure theorem we know that each lamination consists of a finite disjoint union of isolated geodesics and minimal sublaminations. Thus it suffices to show that each minimal lamination

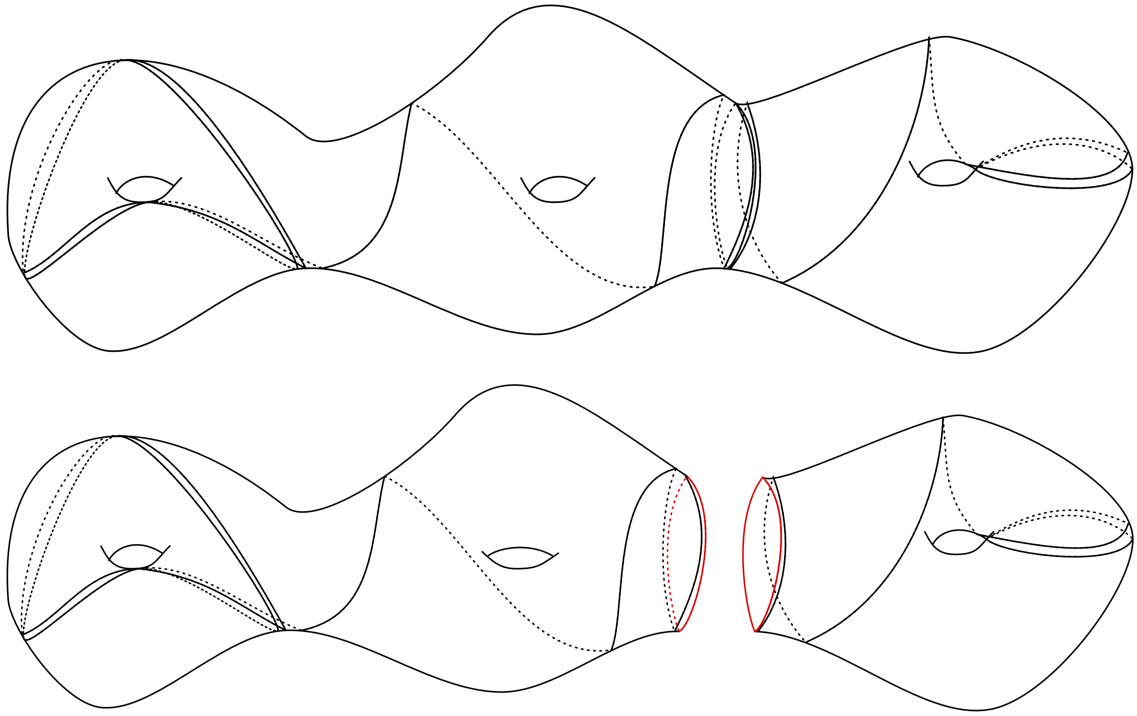


Figure 2.5: A non-minimal lamination Λ which is cut along a minimal sublamination $\lambda \subset \Lambda$ drawn red in the picture. The lamination consists of three closed simple geodesics and two infinite ones spiraling into those

can be approximated by simple closed geodesics. The technique used to approximate minimal geodesics is very similiar to the one we will use later for approximating certain geodesics. It uses quasi-geodesics and thus the next chapter will be about some properties of these curves.

2.1.4 Quasi-geodesics

A quasi-geodesic is, as the name suggests, a curve which is nearly a geodesic in some sense. A complete geodesic $\gamma : \mathbb{R} \longrightarrow M$ in the interior of a surface M can be parametrized by arclength. In this case γ is an isometric embedding of $\mathbb{R}$ into M. Since there is a notion of quasi-isometries we can use this to define a quasi-geodesic.

Definition 2.22. A quasi-geodesic is a curve $\gamma : I \longrightarrow M$ on a hyperbolic surface M, parametrized by arclength, if there exists a constant $k > 0$ such that

$$d_{\text{hyp}}(\gamma(s), \gamma(t)) > k|s - t|$$

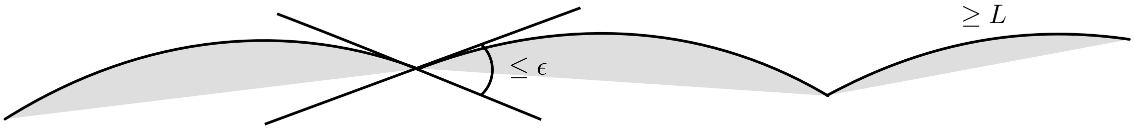
for all $s, t \in I$.

Example 2.23. The most important example for us will be a polygon path, i.e. a piecewise geodesic curve whose geodesic pieces have length at least L and which bend at most a sufficiently small angle θ, see Figure 2.6a. We will later construct such broken geodesics because they can be approximated by simple geodesics in a uniform way.

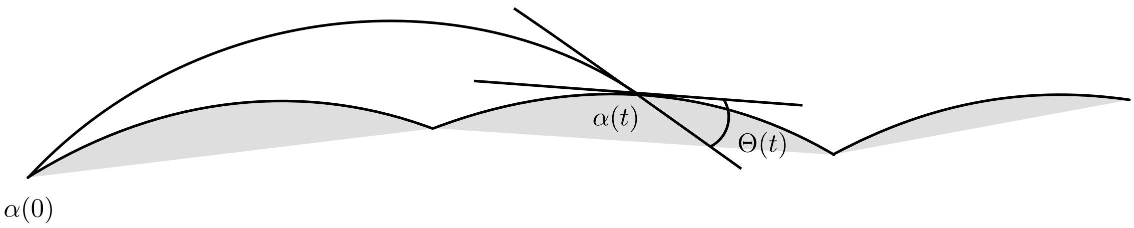
Theorem 2.24. *Fix $\epsilon > 0$. On $\mathbb{H}$, let α be a piecewise geodesic curve whose pieces have length at least ϵ and which has two endpoints. Let β be the geodesic curve joining the endpoints of α. Both curves shall be parametrized by arclength. Then there exists a $\delta > 0$ such that, if the angles between the geodesic segments of α have an angle less than δ then $d_{\text{hyp}}(\alpha(t), \beta(t)) < \epsilon$ for all t.*

This theorem shows that in a bounded distance of a piecewise geodesic curve which is nearly a geodesic there is a geodesic representative. This holds even for hyperbolic surfaces by going to the universal cover, applying the theorem and then projecting back onto the surface.

Proof. The theorem can be shown by investigating the angle $\Theta(t)$ between the geodesic between $\alpha(0)$ and $\alpha(t)$ and the geodesic subarc on which $\alpha(t)$ lies, see Figure 2.6b. First one shows that $\Theta(t) \leq \epsilon$ for all $t \geq 0$ by hyperbolic trigonometry and by choosing δ small enough. Thus $\Theta(t)$ can be made uniformly small. For $x := d_{\text{hyp}}(\alpha(0), \alpha(t))$ one then knows that $\frac{\mathrm{d}x}{\mathrm{d}t}$ is uniformly close to 1 and therefore $d_{\text{hyp}}(\alpha(0), \alpha(t))$ is uniformly close to t. By using triangle inequality one can then show the theorem, see [5] for more details. $\square$



(a) A polygon path in $\mathbb{H}^2$ consisting of geodesic segments of minimal length L and small angles



(b) The function $\Theta(t)$ which is used to show the existence of a geodesic close to a quasigeodesic

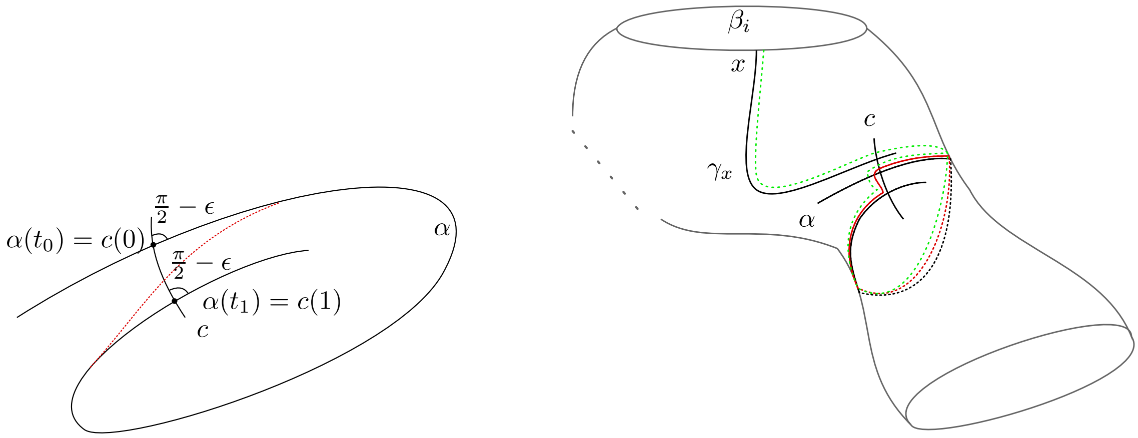
Figure 2.6

In order to use this theorem we now need a way of constructing such polygon paths. Mirzakhani introduces ϵ-good geodesic segments for this purpose.

Definition 2.25. Let $\alpha(t)$ be an arclength-parametrized simple geodesic segment on a hyperbolic surface M and $c : [0, 1] \longrightarrow M$ a smooth arc transverse to α such that $c(0) = \alpha(t_0)$ and $c(1) = \alpha(t_1)$ with $t_0 < t_1$. Then (α, t_0, t_1, c) is called an ϵ-good geodesic segment if

- $l_{\text{hyp}}(c) \leq \epsilon$,
- c is almost perpendicular to α on both ends, i.e. their angle is at most ϵ different from $\frac{\pi}{2}$,
- c meets α in only two points.

The situation is illustrated in Figure 2.7a. Since c is almost perpendicular to both tangent vectors to α they in turn must be almost parallel and the orientations of the pairs $(\alpha'(t_i), c'(i))$ coincide for $i = 0$ and 1. Thus we can define positive and negative good geodesic segments depending on whether the orientation of $(\alpha'(t_i), c'(i))$ coincides with the oriention of the surface or not and this is independent of the parametrization of α.
Now we will suppose that we have some ϵ-good geodesic segment (α, t_0, t_1, c) and construct a complete simple geodesic, see Figure 2.7b. Fix a point x on a boundary component β_i. Then there is a geodesic ray γ_x beginning in x and perpendicular to the boundary. Suppose that it is such that $\alpha \cap \gamma_x = \emptyset$ and $\gamma_x \cap c([0, 1]) \neq \emptyset$ as in the figure. Define $t_2 = \inf\{t \mid \gamma_x(t) \in c([0, 1])\}$. Now denote by $\Psi(\alpha, t_0, t_1, c)$ the simple cosed curve which goes along c from $\alpha(t_1)$ to $\alpha(t_0)$ and then back to $\alpha(t_1)$ along α. By following γ_x from x to $\gamma_x(t_2)$ and then spiralling around $\Psi(\alpha, t_0, t_1, c)$ we obtain a curve η. This can be altered slightly in such a way that η becomes a simple quasi-geodesic by modifying the section of the transverse arc appropriately, see Figure 2.7a. It is then possible to find a geodesic close to η which spirals around $\Psi(\alpha, t_0, t_1, c)$. This is essentially the same technique



(a) A good geodesic segment. In order to make the corresponding curve Ψ quasi-geodesic we replace the transverse arc by some geodesic nearly parallel to the two curves (drawn in red)

(b) A simple geodesic (green) spiraling into some good curve (red) found via quasigeodesics (black with a slight modification)

Figure 2.7

Thurston uses to approximate minimal laminations with the change that we actually want to find a curve spiraling into Ψ instead of just a closed geodesic in the same homotopy class of Ψ.
One more aspect is the following. Suppose we lift the curve γ_x to $\mathbb{H}$. Then we have a polygon path starting at the same point as $\tilde{\gamma}_x$ on $\tilde{\beta}_i$ which bends only in one direction if the angles always have the same sign. This corresponds to ϵ-good segments on the surface of a constant type, i.e. constantly positive or negative. Straigthening of the polygon path gives a geodesic and this geodesic is always on the same side of $\tilde{\gamma}_x$ for constant bending directions. Thus we have control of the side of the geodesic we construct by looking at the sign of the orienation of the ϵ-good geodesic segment.
All these considerations can be summarized in the following lemma.

Lemma 2.26. *Let M be a hyperbolic surface with a boundary component β_i. $x \in \beta_i$, γ_x is the geodesic perpendicular to β_i in x and (α, t_0, t_1, c) is a good geodesic segment such that $\alpha \cap \gamma_x = \emptyset$ and $\gamma_x \cap c([0,1]) \neq \emptyset$. t_2 is defined as before and we can construct η in the same way. Then, for any $\epsilon > 0$ there exist $\delta, L > 0$ such that if (α, t_0, t_1, c) is a δ-good geodesic segment with $L \leq t_1 - t_0$, then η is a simple quasi-geodesic. Let y be the intersection point of the geodesic close to η with β_i. Then $d_{hyp}(x, y) < \epsilon$. Furthermore y lies on the right side of x if and only if (α, t_0, t_1, c) is positive, and vice versa.*

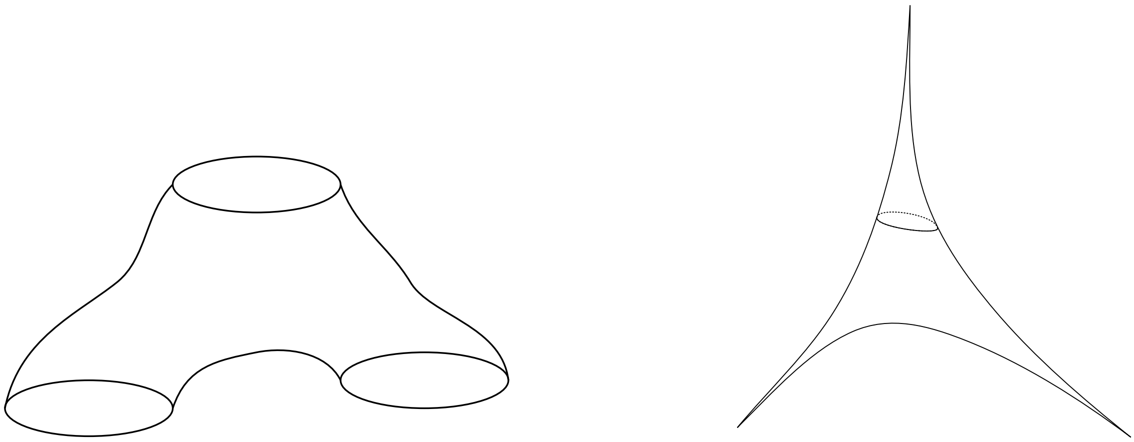
This lemma follows from the earlier considerations and will be used several times in order to find approximating geodesics.

2.1.5 Pairs of pants

The parts that hyperbolic surfaces are built of are hyperbolic pairs of pants. That is why we need to know a lot of facts about them and why we investigate them in this chapter.

Definition 2.27. A hyperbolic pair of pants is a connected two-dimensional orientable surface with genus 0 and 3 boundary components (possibly marked points) together with a hyperbolic metric such that the boundary components are geodesics or the marked points are cusps, respectively. See Figure 2.8.

Lemma 2.28. *A hyperbolic pair of pants can be constructed by gluing two isometric hyperbolic hexagons along alternating sides. Thus, the hyperbolic structure is uniquely determined by the*



(a) A hyperbolic pair of pants

(b) A degenerated hyperbolic pair of pants

Figure 2.8: Hyperbolic pairs of pants

hyperbolic length of the geodesic boundary components (possibly zero) up to isometries, see Figure 2.9.

Proof. See [4]. □

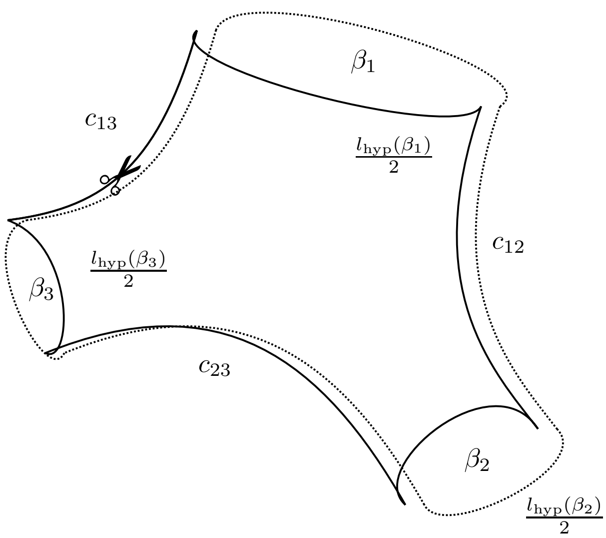


Figure 2.9: A hyperbolic pair of pants can be constructed by gluing two isometric hyperbolic hexagons along alternating sides

Having in mind that in each free homotopy class we have a unique geodesic representative we can deduce the structure of simple geodesics on a pair of pants.

Lemma 2.29. *On a hyperbolic pair of pants there exists exactly three simple complete geodesics joining two different boundary components perpendicularly, exactly three simple complete geodesics joining the same boundary component perpendicularly and exactly twelve simple complete geodesics joining one boundary component perpendicularly and spiraling into another boundary component.*

Proof. See Figure 2.10 and [4]. □

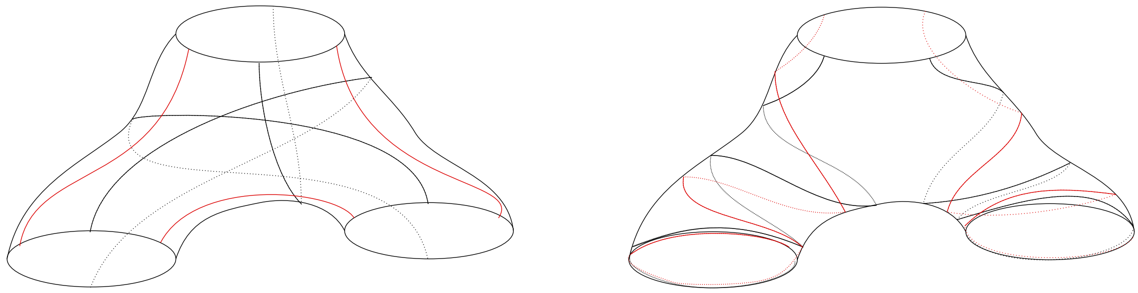


Figure 2.10: In this figure one sees all special geodesics on a hyperbolic pair of pants. The geodesics spiraling into other boundary components from different sides exist of course for all three boundaries. Thus there are 12 of them

Remark 2.30. Since these geodesics are fixed by the hyperbolic structure and thus by the three lengths of the boundary components we see that there are corresponding special points on the boundary components where the geodesics meet the boundary. Furthermore, we expect that the distance between these points is determined by the three lengths. The next theorem shows that this is indeed the case.

Theorem 2.31. *Consider a geodesic boundary β_1 of a hyperbolic pair of pants with boundary lengths L_1, L_2 and L_3. Denote the end points of the geodesics spiraling into β_3 by y_1 and y_2 and those spiraling into β_2 by z_1 and z_2 and those of the geodesic meeting β_1 twice perpendicularly w_1 and w_2. See Figure 2.11a for a picture. Define*

$$\begin{aligned}
\mathcal{R}(L_1, L_2, L_3) &= d_{hyp}(y_1, y_2),\\
\mathcal{D}(L_1, L_2, L_3) &= d_{hyp}(y_1, z_1) + d_{hyp}(y_2, z_2),
\end{aligned}$$

where (y_1, y_2) means the part of the geodesic boundary containing w_1 and w_2 and (y_i, z_i) means the part of the geodesic boundary containing w_i, see Figure 2.11b. Then

$$\begin{aligned}
\mathcal{D}(x,y,z) &= \mathcal{D}(x,z,y),\\
\mathcal{R}(x,y,z) + \mathcal{R}(x,z,y) &= x + \mathcal{D}(x,y,z),\\
\mathcal{D}(x,y,z) &= 2\ln\left(\frac{e^{\frac{x}{2}} + e^{\frac{y+z}{2}}}{e^{-\frac{x}{2}} + e^{\frac{y+z}{2}}}\right),\\
\mathcal{R}(x,y,z) &= x - \ln\left(\frac{\cosh(\frac{y}{2}) + \cosh(\frac{x+z}{2})}{\cosh(\frac{y}{2}) + \cosh(\frac{x-z}{2})}\right)\\
\mathcal{D}(x,y,z) + \mathcal{D}(x,-y,z) &= 2\mathcal{R}(x,y,z),
\end{aligned}$$

where the last equaltiy is of course meant for the extension of the functions to negative numbers.

Proof. Look again at Figure 2.11b to see how the functions are defined in terms of which points on the boundary curve. We know that a pair of pants is constructed from gluing two identical hyperbolic hexagons on the alternating sides. Thus, a pair of pants has an involution. It is given by interchanging the two isometric hexagons, see Figure 2.12a. It corresponds to a sort of reflection. There we see on which hexagon we have which points and how they are glued together. Now, interchanging the lengths y and z is the same as interchanging the two hexagons and then rotating by π in order to have the same orientation of the points, see Figure 2.12b, followed by a renaming of y_i and z_i. Thus interchanging y and z is the same as interchaning y_1 and y_2 as well

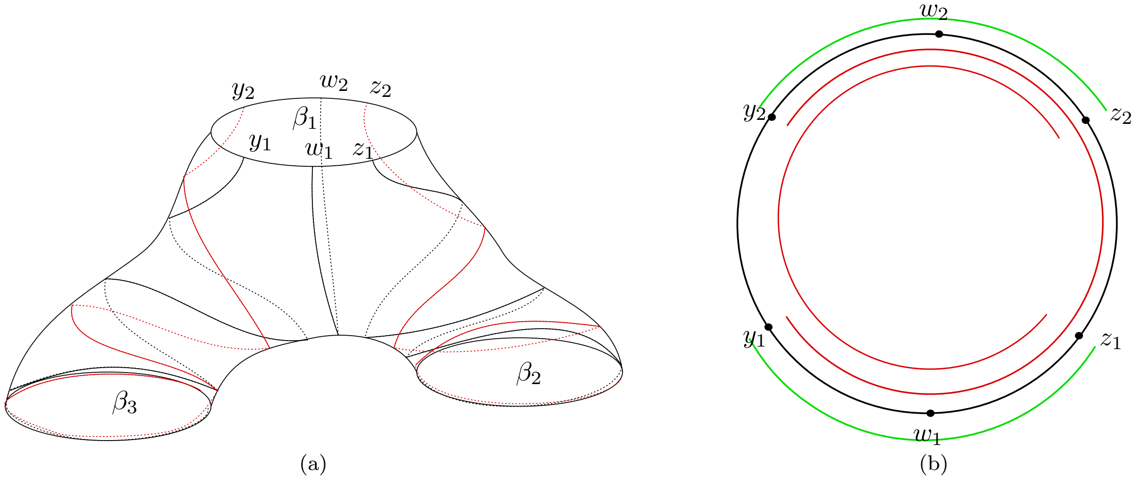


Figure 2.11: The special points on β_1. The green interval refers to the length used in the definition of $\mathcal{D}$ and the red one to the definition of $\mathcal{R}$. Note that the interchange of the two other boundaries switches the two red intervals

as z_1 and z_2, but $\mathcal{D}$ is invariant under this change. Thus

$$\mathcal{D}(x, y, z) = \mathcal{D}(x, z, y).$$

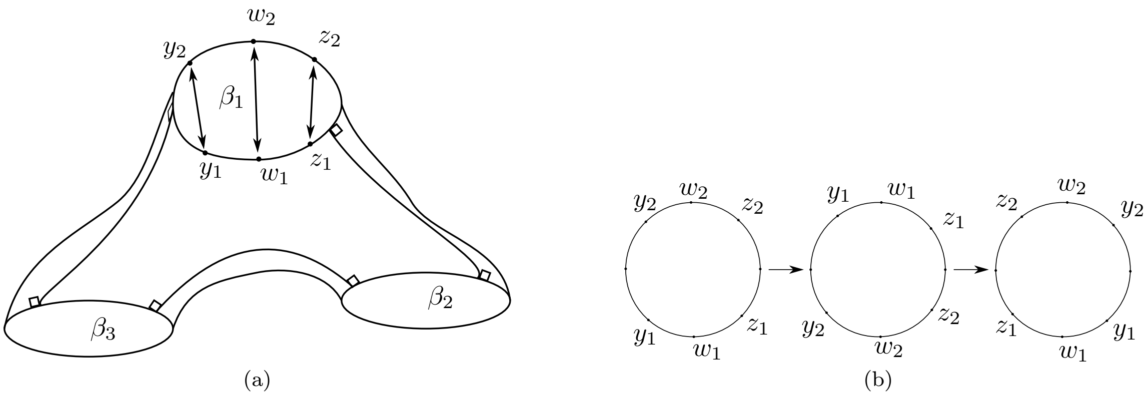


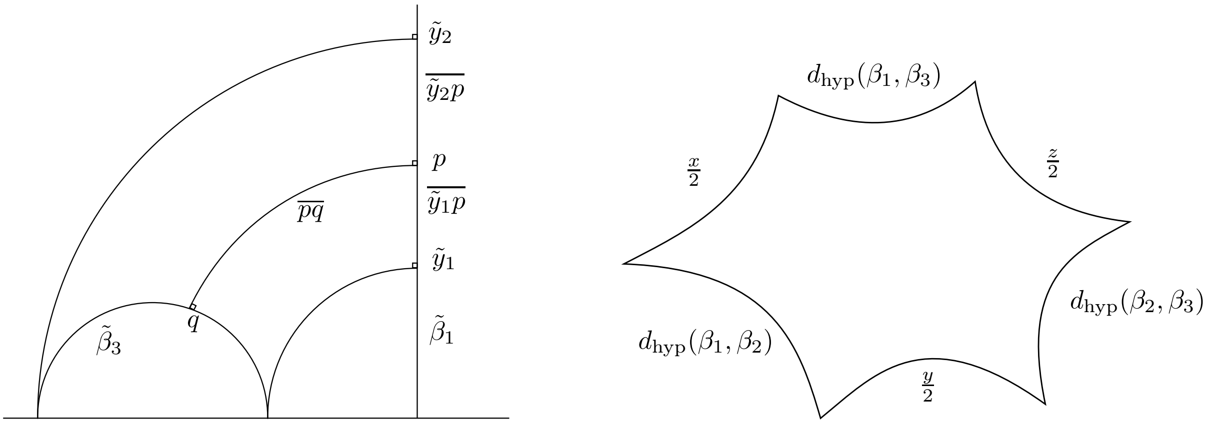
Figure 2.12: The symmetry of a pair of pants. In (b) one first interchanges the two hyperbolic hexagons and then rotates the pair of pants in order to compare the lengths

Again by the reflection property we see that $\mathcal{R}(x, z, y)$ corresponds to taking the length of the other part of the curve, see Figure 2.12b. Now we get immediatly that

$$\mathcal{R}(x, y, z) + \mathcal{R}(x, z, y) = x + \mathcal{D}(x, y, z). \tag{2.1}$$

In order to calculate the functions explicitly we will lift to the universal cover, calculate $\mathcal{R}(x, y, z)$ and then use (2.1) to obtain $\mathcal{D}(x, y, z)$. Choose a lift such that the boundary curve β_1 becomes lifted to a vertical line $\tilde{\beta}_1$ in $\mathbb{H}^2$ and the boundary curve β_3 to $\tilde{\beta}_3$ which is supposed to be the outermost half circle, see Figure 2.13a. Denote by p and q the lifts of the endpoints of the geodesic perpendicular to the two boundary curves.

By Lemma 2.12 we know that for the two tetragons with three perpendicular and one ideal angels



(a) Lift of the pair of pants to $\mathbb{H}^2$

(b) One of the rectangular hyperbolic hexagons obtained by cutting a pair of pants with given boundary lengths along the unique perpendicular geodesics between the boundary components

Figure 2.13

we have

$$\sinh(\overline{pq})\sinh(\overline{\tilde{y}_1 p}) = 1 = \sinh(\overline{pq})\sinh(\overline{\tilde{y}_2 p}).$$

Furthermore, The distance between $\tilde{y}_1$ and $\tilde{y}_2$ is $x - \mathcal{R}(x,y,z)$ by definition. Thus we obtain

$$\mathcal{R}(x,y,z) = x - 2\text{arcsinh}\left(\frac{1}{\sinh \mathrm{d}(\beta_1, \beta_3)}\right),$$

where $d(\beta_1, \beta_3)$ denotes the length of the geodesic perpendicular to the two boundaries β_1 and β_3. Now we cut the pair of pants along these three geodesics to obtain two hyperbolic hexagons with lengths as in Figure 2.13b. Using Lemma 2.11 we get

$$\cosh d(\beta_1, \beta_3) = \frac{\cosh(\frac{y}{2}) + \cosh\frac{z}{2}\cosh\frac{x}{2}}{\sinh\frac{z}{2}\sinh\frac{x}{2}}.$$

Now we continue with the help of arcsinh $x = \ln(x + \sqrt{x^2+1})$ as well as some hyperbolic trigonometric identities and obtain

$$\text{arcsinh}\left(\frac{1}{\sinh\alpha}\right) = \ln\left(\frac{1}{\sinh\alpha} + \sqrt{1 + \frac{1}{\sinh\alpha^2}}\right) = \frac{1}{2}\ln\left(\frac{(1+\cosh\alpha)^2}{\sinh\alpha^2}\right) = \frac{1}{2}\ln\left(\frac{\cosh\alpha + 1}{\cosh\alpha - 1}\right)$$

and therefore

$$\begin{aligned}
\mathcal{R}(x,y,z) &= x - \ln\left(\frac{\cosh \mathrm{d}(\beta_1,\beta_2) + 1}{\cosh \mathrm{d}(\beta_1,\beta_2) - 1}\right) \\
&= x - \ln\left(\frac{\cosh\frac{y}{2} + \cosh\frac{z}{2}\cosh\frac{x}{2} + \sinh\frac{z}{2}\sinh\frac{x}{2}}{\cosh\frac{y}{2} + \cosh\frac{z}{2}\cosh\frac{x}{2} - \sinh\frac{z}{2}\sinh\frac{x}{2}}\right) \\
&= x - \ln\left(\frac{\cosh(\frac{y}{2}) + \cosh(\frac{x+z}{2})}{\cosh(\frac{y}{2}) + \cosh(\frac{x-z}{2})}\right).
\end{aligned}$$

Putting the result for $\mathcal{R}$ into (2.1) we obtain after a lengthy calculation

$$\mathcal{D}(x,y,z) = 2\ln\left(\frac{e^{\frac{x}{2}} + e^{\frac{y+z}{2}}}{e^{-\frac{x}{2}} + e^{\frac{y+z}{2}}}\right).$$

The last identity, which we will later need in order to easier determine the derivatives of these functions, can be shown by a shorter computation

$$\begin{aligned}
\mathcal{D}(x,y,z) + \mathcal{D}(x,-y,z) &= 2\ln\left(\frac{e^{\frac{x}{2}} + e^{\frac{y+z}{2}}}{e^{\frac{-x}{2}} + e^{\frac{y+z}{2}}}\frac{e^{\frac{x}{2}} + e^{\frac{-y+z}{2}}}{e^{\frac{-x}{2}} + e^{\frac{-y+z}{2}}}\right)\\
&= 2\ln\left(\frac{e^{x} + e^{\frac{x-y+z}{2}} + e^{\frac{x+y+z}{2}} + e^{z}}{e^{-x} + e^{\frac{-x+y+z}{2}} + e^{\frac{-x-y+z}{2}} + e^{z}}\right)\\
&= 2\left(x - \ln\left(\frac{e^{-x} + e^{\frac{-x+y+z}{2}} + e^{\frac{-x-y+z}{2}} + e^{z}}{1 + e^{\frac{-x-y+z}{2}} + e^{\frac{-x+y+z}{2}} + e^{z-x}}\right)\right)\\
&= 2\left(x - \ln\left(\frac{e^{\frac{-x-z}{2}} + e^{\frac{y}{2}} + e^{\frac{-y}{2}} + e^{\frac{x+z}{2}}}{e^{\frac{x-z}{2}} + e^{\frac{-y}{2}} + e^{\frac{y}{2}} + e^{\frac{z-x}{2}}}\right)\right)\\
&= 2\left(x - \ln\left(\frac{\cosh\left(\frac{y}{2}\right) + \cosh\left(\frac{x+z}{2}\right)}{\cosh\left(\frac{y}{2}\right) + \cosh\left(\frac{x-z}{2}\right)}\right)\right)\\
&= 2\mathcal{R}(x,y,z).
\end{aligned}$$

□

We will at some point need the existence of special pairs of pants embedded in some hyperbolic surface. This is guaranteed by the following lemma.

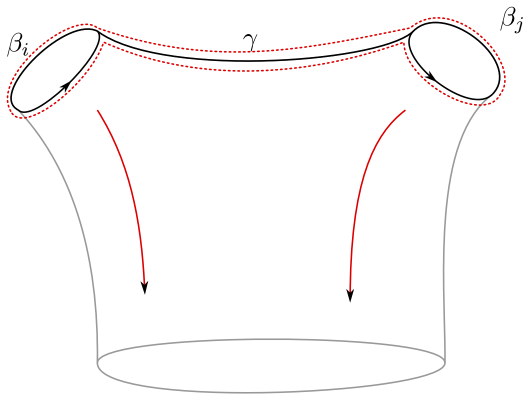
Lemma 2.32. *Let M be a hyperbolic surface with boundary. If γ is a geodesic perpendicular to one boundary on both ends then there exists a unique embedded pair of pants with geodesic boundary such that γ is the special geodesic of this kind on this pair of pants. The same holds if γ is a geodesic perpendicular to two different boundary components.*

Proof. Call the boundary component β. In the first case γ meets β in two points, dividing β into two halfs β_1 and β_2. The concetination of γ and β_i gives an essential simple curve on the surface for both $i = 1$ and 2 which are homotopic to a unique geodesic. Therefore one obtains a unqiue embedded pair of pants, see Figure 2.14b.
In the second case, call the boundaries β_i and β_j. Orient γ in some way and orient the boundary curves with respect to an orienation of the surface. Then consider the concetination of β_i, γ, β_j and γ, see Figure 2.14a. It is an essential simple curve on the surface which is again homotopic to a unique geodesic. Thus one obtains again a unique embedded hyperbolic pair of pants. □

2.2 Structure of the set E_i

2.2.1 Outline of the proof

In this section we will investigate the point set topology of the set of points on a fixed boundary geodesic whose geodesic rays perpendicular to this boundary component are simple. We will see that its structure is such that we can write the length of the boundary component in terms of lengths of intervals between special points in this set. By summing over these points we obtain an equation relating boundary lengths and lengths of good geodesics on the hyperbolic surface. Call the boundary components of the hyperbolic surface X β_i for $i = 1, \ldots, n$. For a point $x \in \beta_i$ denote again by γ_x the geodesic ray perpendicular to β_i through the point x.



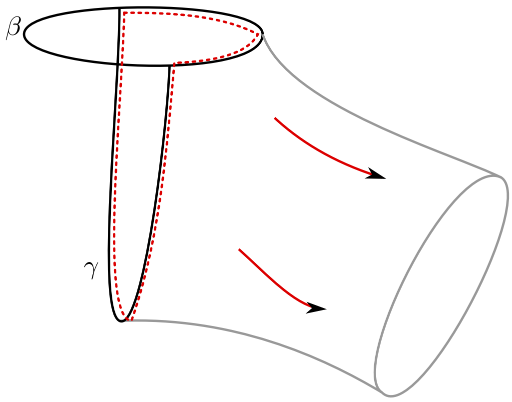


Figure 2.14: The curves constructed to determine the embedded pair of pants

Definition 2.33. The set $E_i \subset \beta_i$ is defined to be the set of all $x \in \beta_i$ such that γ_x is simple. The set E is defined to be the point set of all complete simple geodesics on M, thus $E_i = E \cap \beta_i$.

Theorem 2.34. *Suppose γ_x is a simple complete geodesic on a hyperbolic surface M with one end x perpendicular to some boundary geodesic. Then the other end is either*

1. *perpendicular to some boundary geodesic (possibly the same) or goes up a cusp or*
2. *spirals into some minimal lamination (possibly a closed simple geodesic or a boundary geodesic).*

Proof. See [23]. □

In the next three sections we will investigate the point set topology of the set E_i. Afterwards we will combine these results in order to prove the McShane identity (Theorem 2.44).

2.2.2 Isolated points in E_i

The isolated points in the set $E_i \subset \beta_i$ correspond to endpoints of geodesics perpendicular to β_i which approach some boundary component (either meeting it perpendicularly or spiraling into it). That is, we want to show the following lemma:

Lemma 2.35. *If for $x \in E_i$ the other end of γ_x approaches a boundary component then x is isolated in E_i.*

Proof. *Case 1 – geodesic meets same boundary component*
Without loss of generality we pick β_1 as the boundary under investigation. The first case is the case of a geodesic γ meeting perpendicular the same boundary component. Call the intersection points of this geodesic with the chosen boundary x_1 and x_2. From Lemma 2.32 we know that there exists a unique pair of pants Σ containing γ. On Σ we see again the existence of the usual special geodesics as in Figure 2.10. Changing the notation slightly, see Fig. 2.15, we see

$$x_1 \in E_1 \cap (y_1, y_2),$$
$$x_2 \in E_1 \cap (z_1, z_2).$$

The idea is now to show that $E_1 \cap (y_1, y_2) = \{x_1\}$ and similiary for x_2.
Therefore, let us consider some $z \in E_1$ with $z \notin \{x_1, x_2, y_1, y_2, z_1, z_2\}$. Furthermore, let γ_z be the geodesic ray through z perpendicular to β_1. From the geometry of pairs of pants we know that this simple geodesic ray must leave Σ, therefore it meets $\alpha_1 \cup \alpha_2$. Let us call the boundary it meets first α. Now we lift to the universal cover $\mathbb{H}$ such that the lift $\tilde{\beta}$ of β_1 is a vertical line through

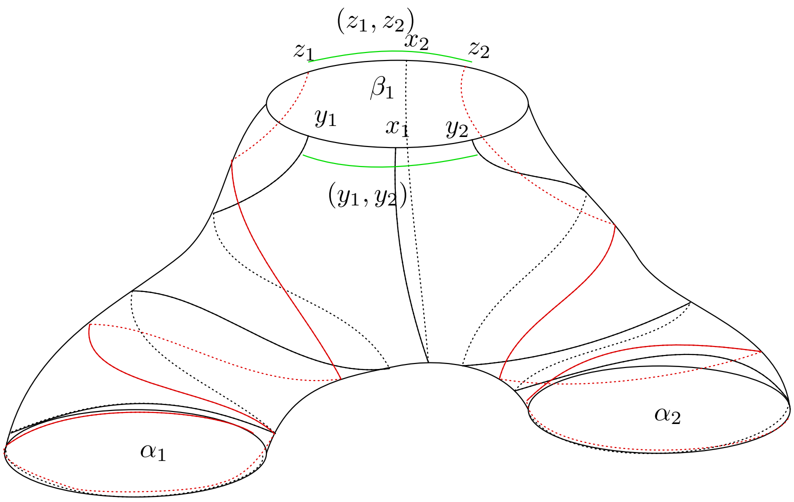


Figure 2.15: The notation for the proof of Lemma 2.35. The green intervals are denoted by $(\cdot, \cdot)$

0 and ∞. Since γ_z meets β_1 perpendicularly there exists a lift $\tilde{\gamma}_z$ to a circle perpendicular to $\tilde{\beta}$, where the intersection point $\tilde{z}$ is a lift of z. By the same reasoning there exists a lift $\tilde{\alpha}$ of α to a circle perpendicular to $\tilde{\gamma}_z$. Since α and β_1 are disjoint and because Γ acts properly discontinous on $\mathbb{H}$ there exists an outermost such lift of α, which we choose to be $\tilde{\alpha}$. Now the situation looks as in Figure 2.16.

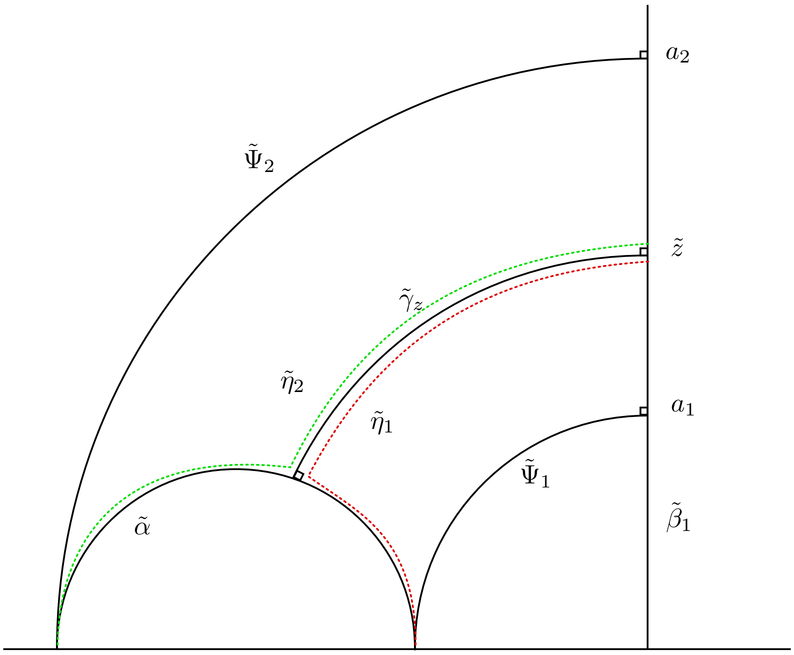
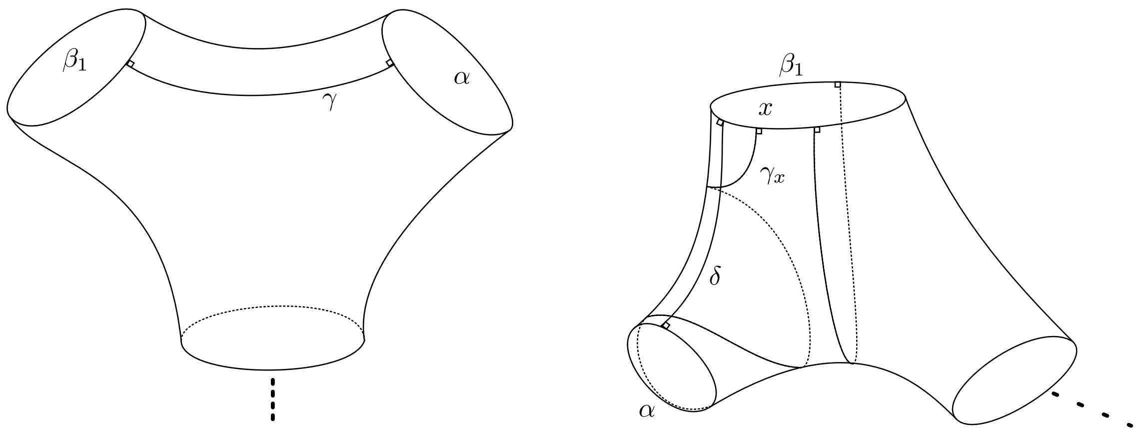


Figure 2.16: Lift to $\mathbb{H}^2$. The red and the green concetinations of curves give $\tilde{\eta}_1$ and $\tilde{\eta}_2$, respectively

Because $\tilde{\alpha}$ is the outermost lift of α meeting $\tilde{\gamma}_z$ and because γ_z as well as α are simple, $\eta_i := \pi(\tilde{\eta}_i)$, where $\tilde{\eta}_i$ is defined as in Figure 2.16, are simple, infinite and piecewise geodesic rays on Σ. Since $\tilde{\eta}_i$ and $\tilde{\Psi}_i$ are homotopic, the projections Ψ_1 and Ψ_2 are complete simple geodesics on Σ spiralling into α. Therefore the points a_1 and a_2 are preimages of z_1 and y_1. Since the geodesic γ does not meet α the endpoints of its lifts are outside the line segment $[a_1, a_2]$ which implies $z \in [y_1, z_1]$ on the surface. This means that the only point in $E_1 \cap (y_1, y_2)$ is indeed x_1 and analogously x_2 is the only point in $E_1 \cap (z_1, z_2)$. Therefore x_1 and x_2 are isolated in E_1.

Case 2 – geodesic meets different boundary component
If the geodesic γ meets another boundary component α perpendicularly we have a situation as in Figure 2.17a in which we know again the existence of such a pair of pants. We lift again to $\mathbb{H}$ such that $\tilde{\beta}_1$ is a vertical line, $\tilde{\gamma}$ is a half circle perpendicular to $\tilde{\beta}_1$ and $\tilde{\alpha}$ is the outermost lift of α perpendicular to $\tilde{\gamma}$. By the same reasoning as in the first case we see that the two half circles touching $\tilde{\alpha}$ asymptotically are lifts of the geodesics spiraling into α on the surface. Since the geodesic meeting two geodesics perpendicular in $\mathbb{H}$ is unique all half circles through points between a_1 and a_2 intersect $\tilde{\alpha}$ non-perpendicularly which means that in (a_1, a_2) there are no lifts of points in E_1 except $\tilde{x}_1$. Thus, it is isolated.



(a) γ meets another boundary component perpendicularly

(b) γ spirals into another boundary component

Figure 2.17

Case 3 – geodesic spirals into boundary component
Now we suppose that γ is a geodesic meting β_1 perpendicularly and spiraling into a boundary component α. We know that there exists a unique geodesic δ perpendicular to both α and β_1, that it is isolated as in case 2 and that there exists a unique pair of pants such that δ lies in this pair of pants, see Figure 2.17b. Furthermore on this pair of pants there exists a unique geodesic perpendicular to β_1 on both ends. By case 1 we know that it is isolated and that the closest points in E_1 are those whose geodesics spiral into the boundaries of the pair of pants. Thus, the endpoint of γ on β_1 is isolated, too. □

2.2.3 Boundary points in E_i

Now we turn to the case of points lying on geodesics whose other end point spirals into a non-peripheral closed geodesic. Those points correspond to boundary points of E_i which means that they are isolated from one side. So we want to show the following lemma:

Lemma 2.36. *If γ is a geodesic on the surface M with one end perpendicular to some chosen boundary geodesic β_1 and with the other end spiraling into some closed non-peripheral geodesic $\Omega(\gamma) =: \lambda$ then there exists a sequence of points $(x_i) \subset E_1$ converging to x from one side, where x is the end point of γ on β_1.*

Proof. The idea of the proof is to take a geodesic that intersects λ exactly once and then Dehn twist around λ. As we will see this determines a sequence of simple geodesics perpendicular to β whose endpoints actually converge to the endpoint of the geodesic that spirals into λ. We will first construct the starting point of the sequence.

Part 1 – Construction of γ_1

There are again two cases:

Case 1 – λ *is separating*
Since λ was essential the connected component of the cut surface without β_1 is either a pair of pants or something more complicated which contains essential closed simple curves.
In the case, that the remaining part is a pair of pants one considers a collar neighborhood of width less than the injectivity radius of the tick part of the surface. Take the curve γ, follow it until you reach the collar neighborhood and then move along a geodesic perpendicular to λ until you arrive at λ. Then one chooses a geodesic on the pair of pants perpendicular to one of the boundaries and λ and connects the two curves via a segment on λ. Then one straightens this curve to give a geodesic. It intersects λ exactly once and is simple and perpendicular to β_1, see Figure 2.18a.

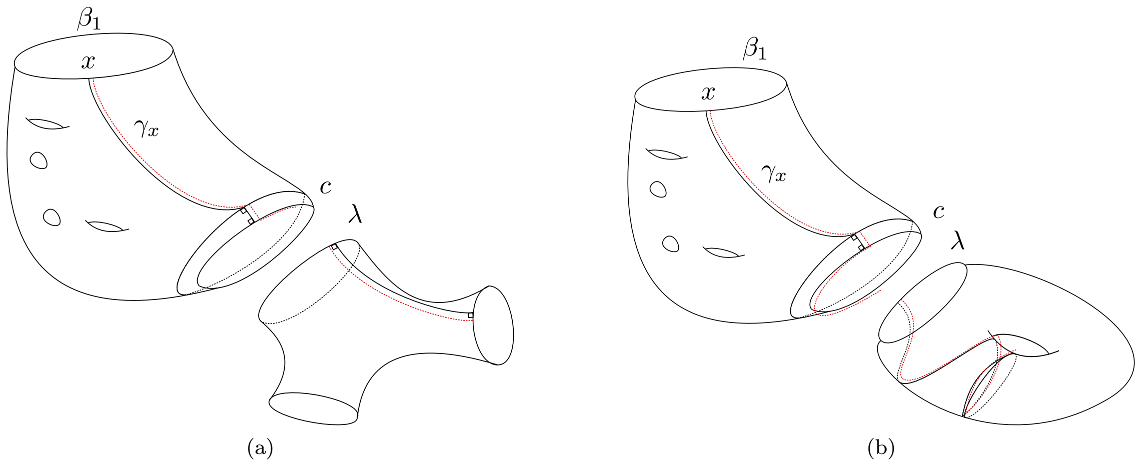


Figure 2.18: The choice of the curve γ_1 in the cases that the surface obtained by cutting along λ is disconnected

Now, if the other part is something containing essential closed simple curves one alters the construction only at the end. Instead of choosing a geodesic perpendicular to another boundary component one chooses a geodesic spiraling into a good curve on the cut surface and which is perpendicular to λ. Then one connects it to the first part on λ and straightens it, see Figure 2.18b.

Case 2 – λ *is non-separating*
Cut M along λ to obtain a surface like in Figure 2.19. As in the first case one follows the geodesic γ until one reaches a close collar neighborhood of one of the images of λ on the cut surface. Then one goes along a geodesic perpendicular to λ until λ. Furthermore one picks a curve perpendicular to β_1 and the other image of λ which does not intersect the first curve. By connecting these two curves on λ and straightening one obtains a simple geodesic perpendicular to β_1 on both ends which intersects λ exactly once.

Part 2 – Dehn-twisting

Case 1 – λ *non-seperating*
Now we lift to $\mathbb{H}^2$ such that $\tilde{\beta}_1$ is a vertical line. Choose a lift of γ which is a half circle perpendicular to $\tilde{\beta}_1$. Choose as a lift $\tilde{\lambda}$ of λ the uppermost half circle touching $\tilde{\gamma}$. In the interior of $\tilde{\lambda}$ there exists another uppermost lift of β_1 denoted by $\tilde{\beta}_1'$, which is disjoint from $\tilde{\lambda}$ because the two curves are disjoint on the surface. Now γ_1 is a curve perpendicular to β_1 on both ends and intersecting

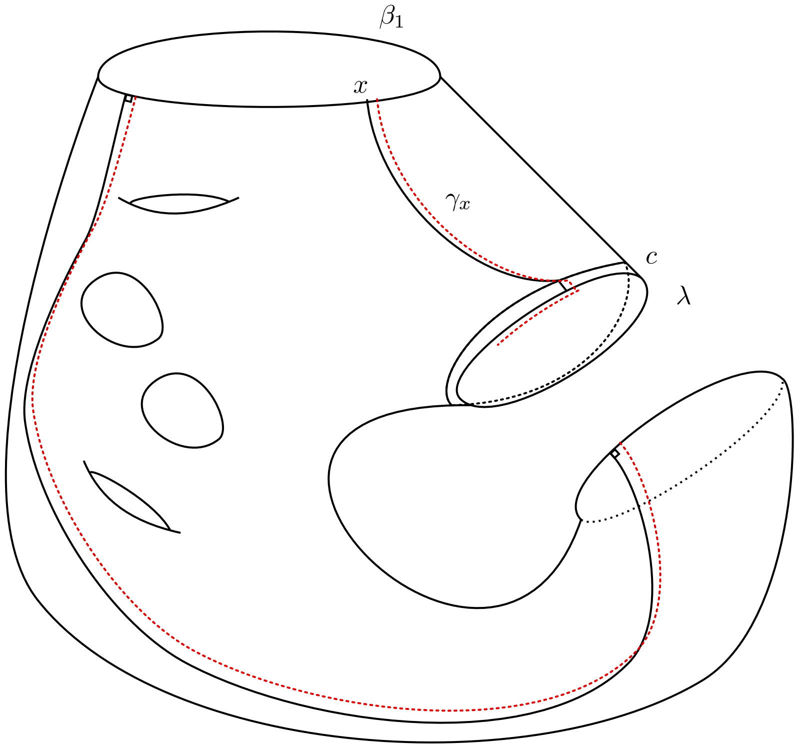


Figure 2.19: The choice of γ_1 if λ is non-seperating

λ exactly once. Thus there exists a lift $\tilde{\gamma}_1$ perpendicular to $\tilde{\beta}_1$ and $\tilde{\beta}_1'$ intersecting $\tilde{\lambda}$ once, see Figure 2.20. Call $A \in \Gamma$ the simple hyperbolic element represented by the axis $\tilde{\lambda}$ with attracting fix point the end touching the curve $\tilde{\gamma}$. It exists in Γ because $\tilde{\lambda}$ projects to a simple closed geodesic on the surface. Furthermore, let z denote the intersection point of $\tilde{\gamma}_1$ and $\tilde{\lambda}$. Consider the sequence of curves $\tilde{\gamma}_n$ obtained by following $\tilde{\gamma}_1$ from $\tilde{\beta}_1$ to z, then to $A^{n-1}z$ and on the other side on the arc defined by the arc perpendicular to $A^{n-1}(\tilde{\beta}_1') =: \tilde{\beta}_1''$ through $A^{n-1}z$, see Figure 2.20.
By straightening the image of this curve on the surface we obtain a new curve γ_n which is simple because the constructed curve was simple and whose endpoints are perpendicular to β_1 after straightening. Since the endpoint and the intersection point on $\tilde{\lambda}$ of the lift are fixed it is covered by the arc used to construct it at the end, see again Figure 2.20. Thus we see that the distance on the cover $\tilde{\beta}_1$ between $\tilde{\gamma}_n$ and $\tilde{\gamma}$ decreases and converges to zero (since $A^{n-1}z$ converges to the attracting fix point which is the endpoint of $\tilde{\gamma}$). Thus the image of $\tilde{\gamma}_n$ on the surface is a sequence of simple geodesics with ends perpendicular to β_1 whose one end x_n converges to an endpoint of the geodesic γ which spirals into a good curve on the surface. Following the same arguments on the surface we see that the construction actually coincides with Dehn-twisting the curve γ_1 around λ.

Case 2 – λ seperating
Here the idea is the same, however, we need to replace $\tilde{\beta}_1'$ by a lift of the curve δ which is the good curve on the surface which was chosen to be the geodesic that γ_1 spirals into. We again choose the hyperbolic element A moving the axis $\tilde{\lambda}$ towards $\tilde{\gamma}$. This now maps the lower part of $\tilde{\gamma}_1$ and $\tilde{\delta}$ closer to the attracting fix point such that the curve $\tilde{\gamma}_n$ obtained as above is covered by an arc closer to $\tilde{\gamma}$. In this way we again obtain a sequence of curves whose endpoints x_n on β_1 converge to x. Furthermore we see again, that on the surface this corresponds to Dehn twisting the curve γ_1 n-times around λ such that it is again simple, perpendicular to β_1 and spirals into δ. □

Remark 2.37. Note that this proof works only from one side of the point $x \in E_i$ as γ_x spirals into

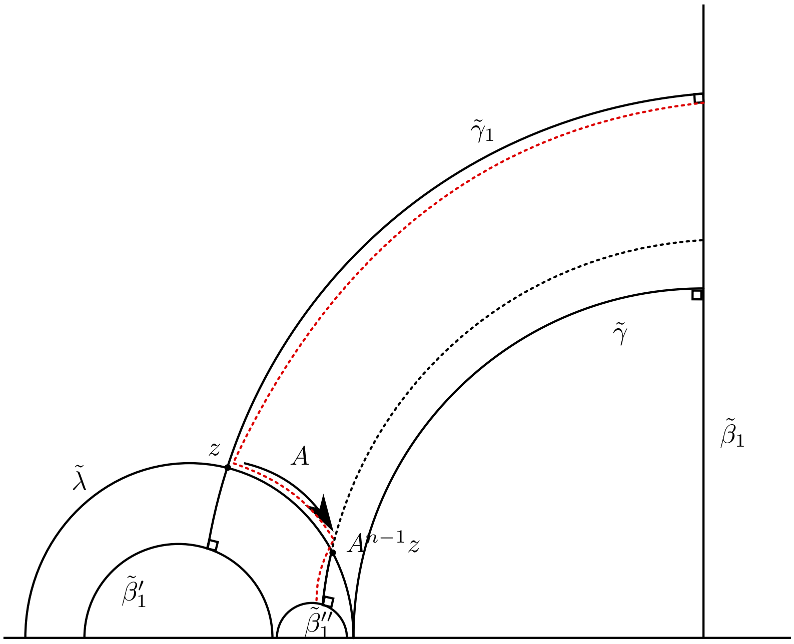


Figure 2.20: The lift to $\mathbb{H}^2$

the simple closed geodesic from one side and thus all the lifts of λ which are ultraparallel to $\tilde{\gamma}$ are on one side of $\tilde{\gamma}$ only.

2.2.4 Other points in E_i

The last statement about points in E_i we want to make is about those geodesics whose other end spirals into a lamination which is not a simple closed curve. We will use the technics from Sect. 2.1.4 in order to show that endpoints of those geodesics can be approximated by points in E_i from both sides:

Lemma 2.38. *If for $x \in E_i$ $\Omega(\gamma_x)$ is not a simple closed curve then x is neither isolated nor a boundary point in E_i.*

Proof. The idea of the proof is to use $\gamma := \gamma_x$ as well as $\lambda := \Omega(\gamma_x)$ to construct a quasi-geodesic and then use Lemma 2.26 to find a close geodesic whose endpoint on the chosen boundary β is arbitrarily close to the endpoint of γ. In this proof we use a couple of statements about minimal laminations.

Part 1 – Find a good geodesic segment in λ
Since λ is a minimal lamination and unequal to a closed curve it consists of uncountably many leaves and each leaf is dense in λ. The last statement holds because the closure of a geodesic is a lamination and since λ does not contain any sublamination the closure is equal to λ. We claim now that

Claim: For all $\epsilon > 0$ and $L > 0$ there exists $t > 0$, an arc c transverse to λ and $y \in c \cap \lambda$ such that

- $(\Phi_y, 0, t, c)$ is a positiv ϵ-good geodesic segment inside λ, where Φ_y is the leaf of λ through y
- $|t| > L$
- $y = \Phi_y(0)$ and $\Phi_y(t)$ are not boundary points in $\lambda \cap c$

The first part then consists in proving this claim. The situation we want to have is pictured in Figure 2.21a. To begin with pick an arbitrary point $y \in \lambda$ and choose a transverse almost perpendicular arc c' such that $\lambda \cap c' \neq \emptyset$. Because λ is not a simple closed curve this set is uncountable but has only countably many boundary points (see Lemma 2.18). Therefore, there are only countably many leaves of λ with some boundary point of $\lambda \cap c'$ on them. Thus it is possible to choose a point $x \in \lambda \cap c'$ such that $\Phi_x \cap c'$ does not contain any boundary points of $\lambda \cap c'$.

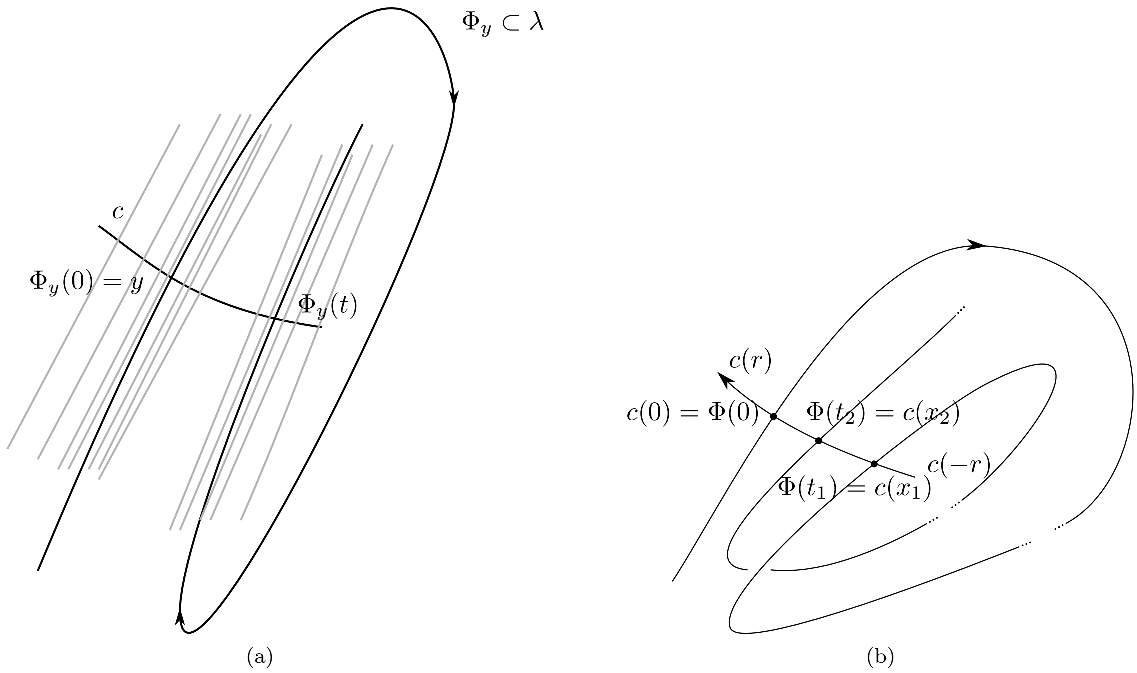


Figure 2.21: The good geodesic segment we want to construct

Now call $\Phi := \Phi_x$, parametrize it by arc-length and choose a transvere subarc $c \subset c'$ such that for $r > 0$ $c : [-r, r] \longrightarrow M$ and for $a, b \in \mathbb{R}$ such that for all $\Phi(a) \neq \Phi(b) \in c([-r, r])$ one has $|a - b| > L$. This means that the distance between the intersection points of $c \cap \Phi$ measured along Φ is at least L. Such a subarc exists because we can pick sufficiently small r and because there are only finitely many points on $c \cap \Phi$ whose distance along Φ is smaller than some finite value. See Figure 2.21b for a picture of the chosen parametrization and the important points. Now choose an orientation of c such that $(\Phi'(0), c'(0))$ has the same orientation as the underlying surface M. The idea is now that Φ is long enough to get within distance ϵ to each point of λ and that no point in $c \cap \Phi$ is a boundary point or isolated, thus every point can be approximated by points in Φ from each side. Therefore the following two times are well defined:

$$t_1 := \inf\{t > 0 | \Phi(t) \in c([-r, 0])\},$$

with x_1 such that $\Phi(t_1) = c(x_1)$ and

$$t_2 := \inf\{t > t_1 | \Phi(t) \in c((x_1, 0))\},$$

where we again define x_2 correspondingly, i.e. $\Phi(t_2) = c(x_2)$.
First of all, there are two possibilities for the orienation of $(\Phi'(t_1), c'(x_1))$. If the orientation is positive then $(\Phi, 0, t_1, c)$ is an ϵ-good geodesic segment (choose r small enough). If the directions are opposite there are two possibilities for the orienation of $(\Phi'(t_2), c'(x_2))$. In either case we obtain ϵ-good geodesic segments, either (Φ, t_1, t_2, c) or $(\Phi, 0, t_2, c)$, see Figure 2.22. By choice of

the subarc c we have $\min\{|t_1 - t_2|, t_1, t_2\} > L$.

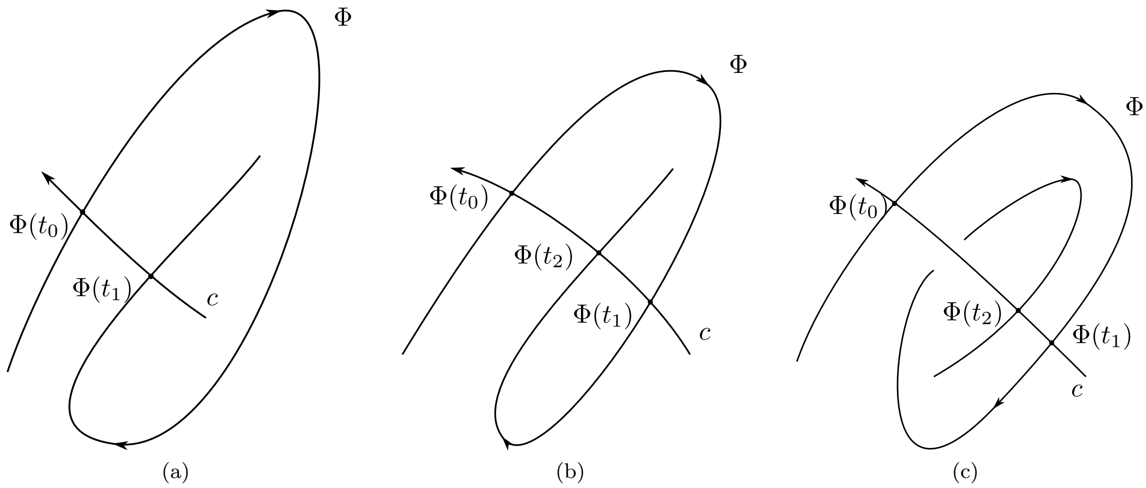


Figure 2.22: The possible ϵ-good geodesic segments

Part 2 – Construction of the geodesic approximating γ
As shown in Figure 2.23 we use this geodesic segment in order to construct a geodesic polygon path from $x \in \beta$ spiraling to $\Phi \in \lambda$ which is a quasi-geodesic. Then we can find a complete simple geodesic lying in a bounded distance from this quasi-geodesic which means in particular that we find a point $x' \in \beta$ in a bounded distance from x. Furthermore the construction will be such that we can arrange those points to lie on any of the two sides of x such that it will be approximated from two sides, excluding the possibility of boundary points.
First of all, γ spirals to λ, thus all points in Φ are approximated by points in γ and therefore, γ intersects c. We define the time t_0 by $t_0 := \inf\{t|\gamma(t) \in c([x_1, x_2])$. Just as we did before, Ψ shall denote the closed simple curve around the geodesic segment Φ and the transverse arc c. Then we find a complete simple curve from x following γ until $\gamma(t_0)$ and then spiraling into Ψ. Now, since γ spirals to λ its tangent vectors become almost parallel to Ψ' at the intersection points of c with γ. Since the geodesic segment through a point of c which spirals into Ψ becomes almost parallel to Ψ, too, we can make c shorter such that the constructed complete simple curve from x spiraling into Ψ is a geodesic polygon path and a quasi-geodesic. Thus, by the general result about quasi-geodesics in 2.26 we find a complete simple geodesic in a bounded distance from γ thus approximating x. Choosing ϵ small enough one can thus approximate x arbitrarily close by points in E_i. Since one wants to use the same leaf when making c shorter one needs here that the intersection points are no boundary points. By choosing the orientation of Φ in the construction of the ϵ-good geodesic segment we can change the orienation of this segment appropriately. The resulting polygon path will then always be at the same side of the geodesic and thus the approximating sequence in β will always be on the same side. Thus by changing the orientation we obtain an approximation from the other side, showing that x is neither isolated nor a boundary point. $\square$

2.3 Proof of the McShane identity

Let us summarize the results of Section 2.2 once more:

Lemma 2.39. *Let M be a surface as before with boundaries $\beta_1, \cdots, \beta_n$, E be the set of all points in M lying on a simple geodesic perpendicular to boundaries and $E_i := E \cap \beta_i$. Then, for any $x \in E_i$ exactly one of the following holds:*

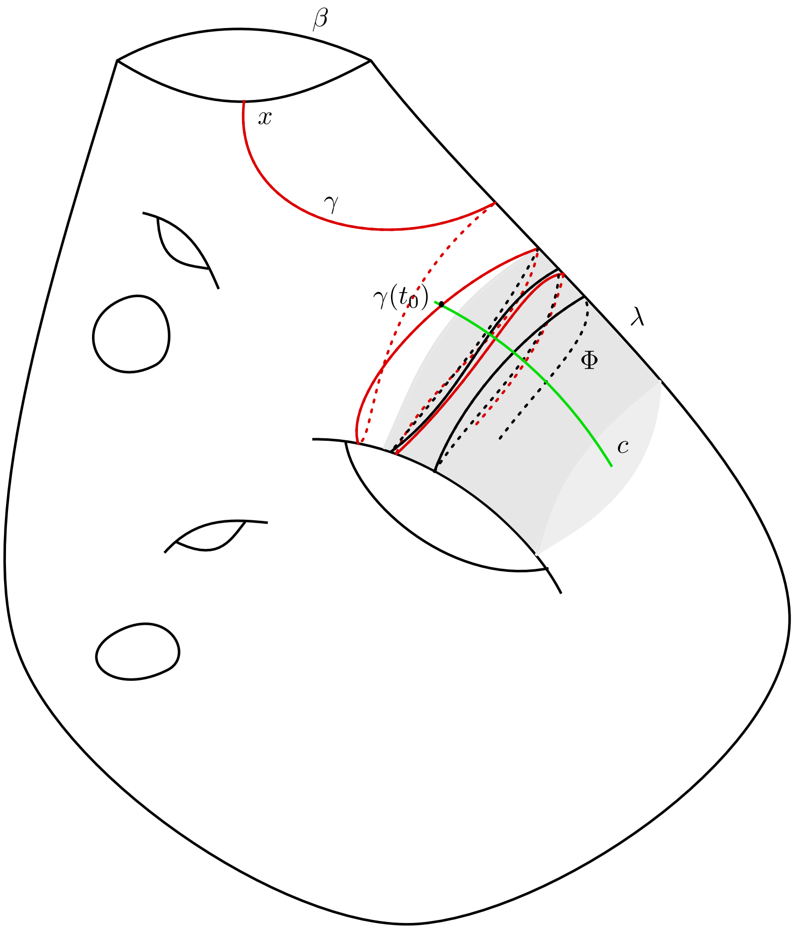


Figure 2.23: The construction of the quasigeodesic. Φ (black) is the dense leaf in λ. The quasigeodesic (red) is constructed by following γ (also red, but only drawn at the beginning) until one meets the green transverse arc c. Then one spirals into Ψ, which is obtained by following Φ and c. The picture is a little bit misleading since the lamination cannot be contained in the cylinder but has to wrap around somewhere else, thus making it possible that Ψ approaches itself again

1. *The other end of γ_x approaches a boundary component and x is isolated in E_i*
2. *$\Omega(\gamma_x)$ is a non-peripheral simple closed curve and x is a boundary point of E_i*
3. *$\Omega(\gamma_x)$ is not a simple closed curve and x is neither isolated nor a boundary point of E_i*

This lemma was proven in the last section. For our further calculations we will need the following result, due to Birman and Series:

Theorem 2.40 (Birman and Series)**.** *On a closed hyperbolic surface M the set E has Hausdorff dimension 1.*

Proof. See [3]. $\square$

We will now use this theorem to show a couple of lemmas.

Lemma 2.41. *For any hyperbolic surface with geodesic boundary the set E_i has one dimensional Hausdorff measure zero.*

Proof. By doubling M along the geodesic boundaries we obtain a closed hyperbolic surface $\overline{M}$. Since the geodesics under consideration are either disjoint from the boundary or meet the boundary perpendicularly they extend to simple geodesics on $\overline{M}$. Thus $E_i \subset E(\overline{M})$, where $E(\overline{M})$ denotes the points lying on simple geodesics on $\overline{M}$. By the result of Birman and Series we have $\mu(E(\overline{M})) = 0$ for the two-dimensional Hausdorff measure μ because $\dim_{\mathrm{H}} E(\overline{M}) = 1$ where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension. Now pick a small collar neighborhood U_{β_i} of $\beta_i \subset E(\overline{M})$. Then $E(\overline{M}) \cap U_{\beta_i}$ is homeomorphic to $E_i \times I$ where I is some interval. Thus we have by standard results for the Hausdorff dimension (see [12])

$$1 = \dim_{\mathrm{H}} E(\overline{M}) \cap U_{\beta_i} = \dim_{\mathrm{H}} E_i \times I \geq \dim_{\mathrm{H}} E_i + \dim_{\mathrm{H}} I = \dim_{\mathrm{H}} E_i + 1$$

and therefore $\dim_{\mathrm{H}} E_i = 0$ which implies that the one dimensional Hausdorff measure of E_i is zero. □

Lemma 2.42. *E_i is homeomorphic to a Cantor set union countably many isolated points.*

Proof. By Theorem 2.39 we know that the isolated points correspond to geodesics approaching some boundary geodesic. Since there are finitely many boundary geodesics and since $\{$continuous curves from β_i to $\beta_j\}/$homotopy is countably infinite we have at most countably many such isolated points. Removing those points we are left with a set E_i', all of whose points are limit points. Because β_i is compact and $E_i' \subset \beta_i$ is closed, E_i' is compact. Furthermore E_i' is closed and has no isolated points, thus it is a perfect set. And since it has measure zero it does not contain any interval and is thus totally disconnected. Therefore, E_i' is homeomorphic to a Cantor set. □

In order to show the McShane identity we will now use the previous results to write the length of a chosen boundary geodesic as a sum over many intervals. Then, those intervals will correspond to embedded pairs of pants which are bound by simple closed geodesics at the boundary or in the interior. The length of the intervals in β_i will then be expressed in terms of the length of the determined simple closed geodesics.
First of all, we will make some definitions on the set E_i. I_i denotes the set of isolated points in E_i, it means those points whose geodesic's other end approaches some boundary component. By the last lemma we know that E_i consists of a Cantor set union countably many isolated points, especially it does not contain any interval but discrete points only. Thus its complement in E_i consists of a union of disjoint intervals. Let H be the set of connected components of $I_i \cup (\beta_i \setminus E_i)$, that is, we put back in the isolated points. Then

$$I_i \cup (\beta_i \setminus E_i) = \bigcup_{h \in H} (a_h, b_h),$$

where a_h and b_h in $E_i \backslash I_i$ are the beginning and end points of the (maximal) interval h, respectively. This means that those points are isolated in E_i from exactly one side, because from one side there is the interval in the complement. Now we relate H to embedded pairs of pants containing β_i.

Lemma 2.43. *There is a $2-1$ correspondence between intervals $h \in H$ and pairs of pants containing β_i.*

Proof. Suppose we are given a pair of pants Σ embedded in M which contains β_i. Then we know that there exist four geodesics perpendicular to β_i which lie in Σ and which spiral to the boundary components of Σ, see Figure 2.10. These geodesics bound two intervals in H because of Lemma 2.39.
Now for the converse suppose we are given an interval $h \in H$. Then γ_{a_h} spirals into a non-peripheral simple closed geodesic $\Omega(\gamma_{a_h})$. Let Σ_h be the unique pair of pants in M containing γ_{a_h} and β_i. Furthermore, let α be the third boundary of this pair of pants. We claim that $\gamma_{b_h} \subset \Sigma_h$.

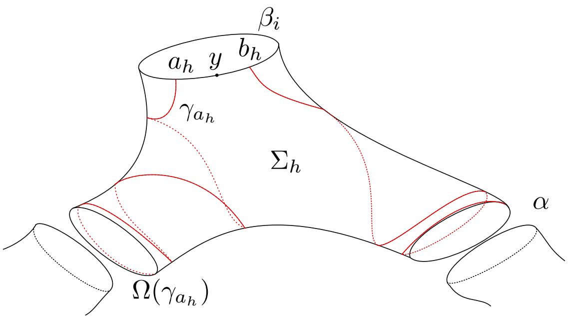
If this is the case then we can associate a unique pair of pants to the interval h with γ_{a_h} and γ_{b_h} being two of the special geodesics on this pair of pants. There are two possibilities for α:

Case 1 – α is non-peripheral

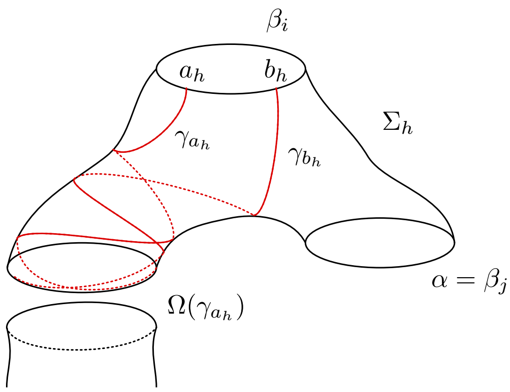
The situation looks as in Figure 2.24a. If we follow a_h along β_i in the direction of the interval the first point in $E_i \setminus I_i$ is y with γ_y spiraling into the geodesic α because of Theorem 2.39. Thus $y = b_h, \gamma_{b_h} \subset \Sigma_h$ and $\Omega(\gamma_{a_h}), \Omega(\gamma_{b_h})$ and β_i bound an embedded pair of pants.

Case 2 – $\alpha = \beta_j$ is a boundary component

In this case, β_i, β_j and $\Omega(\gamma_{a_h})$ bound Σ_h. Now following β_i from a_h into the direction of the interval we see that we pass only isolated points in E_i because α is a boundary geodesic. Thus the next endpoint of a geodesic in $E_i \setminus I_i$ is the point whose geodesic spirals into $\Omega(\gamma_{a_h})$ from the other side, see Figure 2.24b. This means that γ_{b_h} is exactly this geodesic and therefore $\gamma_{b_h} \subset \Sigma_h$ and β_i, β_j and $\Omega(\gamma_{a_h}) = \Omega(\gamma_{b_h})$ bound a pair of pants in M. $\square$



(a)



(b)

Figure 2.24: The two cases for the geodesic α

Having established the correspondence between intervals $h \in H$ and embedded pairs of pants, we can now prove the generalized McShane identity.

Theorem 2.44 (Generalized McShane Identiy). *Let M be a hyperbolic surface with geodesic boundaries of length $L_1, \ldots, L_n$. Then*

$$\sum_{\{\gamma_1,\gamma_2\}\in\mathcal{F}_1} \mathcal{D}(L_1, l_{\gamma_1}(M), l_{\gamma_2}(M)) + \sum_{i=2}^{n} \sum_{\gamma\in\mathcal{F}_{1,i}} \mathcal{R}(L_1, L_i, l_\gamma(M)) = L_1. \tag{2.2}$$

Here, the sets $\mathcal{F}_{i,j}$ and $\mathcal{F}_i$ are defined as follows. $\mathcal{F}_i$ is the set of unordered pairs of simple closed geodesics bounding a pair of pants with β_i and $\mathcal{F}_{i,j}$ is the set of simple closed geodesics bounding a pair of pants with β_i and β_j for $i \neq j$.

Proof. Let us first recall that in the terminology of this section

$$I_i \cup (\beta_i \setminus E_i) = \bigcup_{h\in H} (a_h, b_h).$$

By Lemma 2.41 we see that we can calculate the length of β_i by summing over the lengths of these intervals (a_h, b_h), i.e.

$$L_i = l_{\beta_i}(M) = \sum_{h \in H} d_{\text{hyp}}(a_h, b_h), \tag{2.3}$$

where $d_{\text{hyp}}(a_h, b_h)$ denotes the hyperbolic distance between a_h and b_h measured along β_i in the direction of the interval h. Now, for each $h \in H$ there are two possibilities corresponding to the two cases in the previous correspondence:

Case 1 – $\exists j \neq i$ such that $\beta_i, \beta_j, \Omega(\gamma_{a_h})$ bound a pair of pants in M

In this case, see Figure 2.24b, $\Omega(\gamma_{a_h}) = \Omega(\gamma_{b_h}) =: \gamma$ and we have by the result of Theorem 2.31

$$d_{\text{hyp}}(a_h, b_h) = \mathcal{R}(L_i, L_j, l_\gamma(M)). \tag{2.4}$$

Case 2 – $\Omega(\gamma_{a_h}) \neq \Omega(\gamma_{b_h})$

This means that the two geodesics $\gamma_1 := \Omega(\gamma_{a_h})$ and $\gamma_2 := \Omega(\gamma_{b_h})$ are distinct and bound a pair of pants together with β_i, see Figure 2.24a. Then again by the result of Theorem 2.31, we have

$$\frac{1}{2}\mathcal{D}(L_i, l_{\gamma_1}(M), l_{\gamma_2}(M)) = d_{\text{hyp}}(a_h, b_h). \tag{2.5}$$

Now what remains is to put 2.4 and 2.5 into 2.3. However, as we have already pointed out, given a pair of pants Σ we can associate two intervals h and h' to it, if two geodesics are non-peripheral. Those two intervals have the same hyperbolic length. Thus by summing over the pairs of pants instead of the intervals we undercount and have to include therefore a factor of 2 in the second case. So we obtain

$$\sum_{\{\gamma_1,\gamma_2\} \in \mathcal{F}_1} \mathcal{D}(L_1, l_{\gamma_1}(M), l_{\gamma_2}(M)) + \sum_{i=2}^{n} \sum_{\gamma \in \mathcal{F}_{1,i}} \mathcal{R}(L_1, L_i, l_\gamma(M)) = L_1, \tag{2.6}$$

if we label the chosen boundary geodesic β_i as the first. $\square$

3 Teichmüller-Theory

3.1 Riemann surfaces

Although we have already talked about hyperbolic surfaces we want to investigate general Riemann surfaces a little bit further because we will need a lot statements about them in the sequel.

Definition 3.1. A closed Riemann surface is a connected two-dimensional orientable differentiable manifold without boundary. Its homeomorphism type (as well as its diffeomorphism type) is uniquely determined by its genus g.
A Riemann surface with boundary is a connected two-dimensional orientable differentiable manifold with boundary. Its homeomorphism and diffeomorphism type is uniquely determined by its genus g and the number n of boundary components.
A punctured Riemann surface is a connected two-dimensional orientable differentiable manifold without or with boundary together with a finite set of pairwise disjoint points away from the boundary.

Remark 3.2. We will be interested in compact surfaces only. Thus g and n are always finite in our cases. Furthermore we will not consider the case of a Riemann surface with boundaries and punctures at the same time.

Definition 3.3. Let $\Sigma_{g,n}$ be a Riemann surface of genus g with either n marked points or n boundary components. Then we define the group Diff_+ of $\Sigma_{g,n}$ to be the set of orientation preserving diffeomorphisms $f : \Sigma_{g,n} \longrightarrow \Sigma_{g,n}$ such that each boundary component is setwise fixed and all marked points are pointwise fixed. The group multiplication is given by the composition and the inverse by the inverse map.
Furthermore $\mathrm{Diff}_0 \subset \mathrm{Diff}_+$ is defined to be the set of all orientation preserving diffeomorphisms $f : \Sigma_{g,n} \longrightarrow \Sigma_{g,n}$, fixing boundary components setwise and marked points pointwise, which are isotopic to the identity (i.e. homotopic in the set Diff_+).

Lemma 3.4. *$Diff_0 \subset Diff_+$ is a normal subgroup such that $Mod_{g,n} := Diff_+ \diagup Diff_0$ is a well-defined group. It is finitely generated by so-called Dehn twists around suitable sets of closed simple curves.*

Proof. See [13]. □

Remark 3.5. From the context it will be clear which diffeomorphism group we are talking about, thus the surface will not be noted explicitly.

Definition 3.6. A curve γ on $\Sigma_{g,n}$ is called good if it is closed, simple (i.e. the continous map $S^1 \longrightarrow \Sigma_{g,n}$ is injective) and non-peripheral (i.e. it does not meet any marked points or boundary components). A collection of curves $\{\gamma_i\}_{i\in I}$ is called good if all the individual curves are good and if furthermore the curves are pairwise disjoint and pairwise not freely homotopic.

Lemma 3.7. *Given two good curves γ_1 and γ_2 on $\Sigma_{g,n}$ there exists a diffeomorphism $f \in Diff_+$ such that $f(\gamma_1) = \gamma_2$ if and only if the (unconnected) surfaces obtained by cutting $\Sigma_{g,n}$ along γ_1 and γ_2 are diffeomorphic by a diffeomorphism which fixes the boundaries setwise and the punctures pointwise.*

Proof. See [13]. □

3.2 Definitions of Teichmüller space

3.2.1 Two definitions of Teichmüller space

In this section we will mainly deal with the various definitions of Teichmüller space of closed Riemann surfaces and investigate the relations between them. The reason is that we need lots of concepts such as Fenchel–Nielsen coordinates and the the Weil-Petersson symplectic structure which are defined using different views of Teichmüller space. Thus it is necessary to switch between the most suitable definitions of Teichmüller space for the concrete situation. Furthermore we will actually deal with Teichmüller spaces of bordered Riemann surfaces, however, once the concept of Teichmüller space is understood it is rather easy to extend the concept to Riemann surfaces with boundary. In this work we will distinguish two important definitions, one in terms of almost complex structurs and the other one in terms of marked Riemann surfaces.

Teichmüller space as the set of hyperbolic structures on a Riemann surface

One of the important view points on Teichmüller spaces will be their definition in terms of hyperbolic structures. Thus pick a closed Riemann surface of genus g and call it Σ_g. A marked hyperbolic surface is a pair of a hyperbolic surface X with genus g, i.e. a Riemann surface with genus g and a metric on it whose sectional curvature is minus one, together with a marking $f : \Sigma_g \longrightarrow X$, where f is an orientation preserving diffeomorphism. Two such marked surfaces (X, f) and (Y, g) are defined to be equivalent if and only if $f \circ g^{-1} : Y \longrightarrow X$ is isotopic to a conformal map $\phi : Y \longrightarrow X$. Then we can define

Definition 3.8 (Teichmüller space)**.** The Teichmüller space $\mathcal{T}_g^{(4)}$ is defined to be

$$\mathcal{T}_g^{(1)} := \{\text{marked hyperbolic surfaces}\} \big/ \sim .$$

Since X and Y are diffeomorphic to the model Riemann surface Σ_g we can interprete this as being the set of hyperbolic structures on such a surface modulo some equivalence relation. The section on moduli spaces will clarify this point of view.

Teichmüller space as the set of almost complex structures on a Riemann surface

The second definition of Teichmüller space is very suitable for doing explicit calculations and introducing certain structures on this space. Let $\mathcal{A}$ be the space of almost complex structures on a fixed oriented Riemann surface Σ_g of genus g such that the induced orientation agrees with the given one. This means, $I \in \mathcal{A}$ is a section of the endomorphism bundle of $\mathrm{T}M$ such that $I^2 = -\mathrm{Id}$ at every point. The diffeomorphism group of Σ_g, denoted by Diff_+, acts on $I \in \mathcal{A}$ via push forward. In the same way Diff_0 acts on $\mathcal{A}$. Thus we can make the following definition

Definition 3.9 (Teichmüller space)**.** The Teichmüller space $\mathcal{T}_g^{(2)}$ is defined to be

$$\mathcal{T}_g^{(2)} := \mathcal{A} \big/ \mathrm{Diff}_0.$$

Any other Riemann surface of genus g is diffeomorphic to Σ_g such that we can pullback the almost complex structures to see that the so defined Teichmüller spaces are in a bijective correspondence. However, the identification depends on the chosen diffeomorphism.

3.2.2 Connection between the two definitions

As the nomenclature suggests all these definitions are equivalent. In fact we will frequently change between them because for certain problems one view point is better than the other. For example, in order to define the tautological bundles it is suitable to look at complex structures whereas the hyperbolic viewpoint is prefereable in the actual calculations because the symplectic structure on Teichmüller space is defined in terms of the hyperbolic structure of the Riemann surface.

On one side we have a set of almost complex structures on a fixed Riemann surface and on the other side we have a set of hyperbolic structures together with markings. Suppose we are given $(X, f) \in \mathcal{T}_g^{(1)}$ and want to determine an almost complex structure on Σ_g. Pull back the hyperbolic metric g on X to Σ_g to obtain a hyperbolic metric on Σ_g. This defines an almost complex structure by considering in each tangent space the rotation by $\frac{\pi}{2}$ in the direction of the orientation. It can be shown that this is indeed well defined, i.e. for two equivalent marked surfaces (X, f) and (Y, g) one obtains two almost complex structures which are push-forwards of each other, see [33].
The other direction is a classical example of geometric analysis, because we need to show that there exists a unique solution of a certain partial differential equation on the Riemann surface. Let an almost complex structure J on Σ_g be given. For any point $z \in \Sigma_{g,n}$ choose a neighborhood U_z together with a non-vanishing vector field X on U_z. We define a metric $g_{|U_z}$ by requiring

$$g_{|U_z}(X, X) = 1, g_{|U_z}(X, JX) = 0, g_{|U_z}(JX, JX) = 1.$$

It is easy to see that a change of the vector field only changes the metric by a conformal factor which means that since Σ_g is orientable, it defines a global conformal class of metrics $[g_J]$ on $\Sigma_{g,n}$. Now we choose a unique representative of this conformal class by requiring that the sectional curvature K is constant -1. This corresponds to a partial differential equation $K(\lambda g) = -1$, where g is any representative of $[g_J]$ and λ is the to-be-determined function $\Sigma_g \longrightarrow \mathbb{R}_+$. It can be shown that this has indeed a unique solution $g_{J,-1}$, see [33]. With a bit more work regarding well-definedness we can then define the inverse map from $\mathcal{T}_g^{(2)} \longrightarrow \mathcal{T}_g^{(1)}$ by $[J] \longmapsto [(\Sigma_{g,n}, g_{J,-1}), \mathrm{Id}]$.
Of course, since Teichmüller space is always defined as a quotient one needs to show that the maps are actually well defined and indeed inverses of each other. Since this would take a while we will skip this part and assume that this works out well. Furthermore we have not said anything about continuity or differentiability because we have not even defined a topology on the relevant spaces. This issue will be explained a bit more in 3.3. However, we will not see in which sense the defined mappings are continuous or differentiable with respect to the natural topologies, as this would again take too long.
Anyway we have seen how to identify these spaces and we can conclude by

Lemma 3.10. *The two definitions for Teichmüller space are in a bijective correspondence* $\mathcal{T}_g^{(1)} \simeq \mathcal{T}_g^{(2)}$.

3.2.3 Teichmüller space of bordered Riemann surfaces

In this section we want to extend the definition of Teichmüller space to the case of Riemann surfaces of a specified genus which have boundaries or marked points. The first definition is in terms of hyperbolic structures. Let $L \in \mathbb{R}_+^n$ be a n-tuple of positive real numbers and $\Sigma_{g,n}$ an oriented Riemann surface of genus g and n boundary components denoted by β_i for $i = 1, \ldots, n$.

Definition 3.11. The Teichmüller space of bordered Riemann surfaces of length L, denoted by $\mathcal{T}_{g,n}(L)$ is given by the set of equivalence classes of tuples (X, f), where X is a Riemann surface of genus g with n boundary components and carries a hyperbolic metric such that the boundary components $f(\beta_i)$ are geodesics of length L_i and $f : \Sigma_{g,n} \longrightarrow X$ is an orientation preserving diffeomorphism. Two pairs (X, f) and (Y, g) are identified if $f \circ g^{-1} : Y \longrightarrow X$ is isotopic to a conformal map.

Furthermore we can define the length function for closed simple curves γ on $\Sigma_{g,n}$ by

$$\begin{aligned} l_\gamma : \mathcal{T}_{g,n}(L) &\longrightarrow \mathbb{R}_+ \\ (X, f) &\longmapsto l_{\mathrm{hyp}}(f(\gamma)), \end{aligned}$$

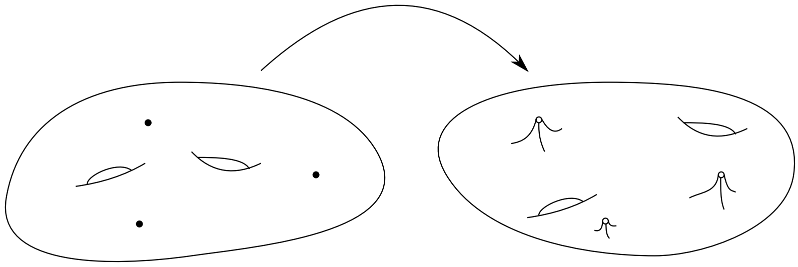


Figure 3.1: The definition of Teichmüller space in terms of hyperbolic structures

where $l_{\mathrm{hyp}}(\gamma)$ denotes the hyperbolic length of the unique geodesic representative in the free homotopy class of $f(\gamma)$. Thus, any element $X \in \mathcal{T}_{g,n}(L)$ satisfies $l_{\beta_i}(X) = L_i$ for all $1 \leq i \leq n$. Later we will be interested on the behaviour of a certain symplectic structure on the Teichmüller space without prescribed lengths of the boundary components in the neighborhood of $L = 0$. Therefore we will now define the Teichmüller space of a punctured Riemann surface. In some sense we may look at this as the limit when the boundary curves become of length zero. Let Σ_g be a closed Riemann surface with genus g.

Definition 3.12. The Teichmüller space of n-punctured Riemann surfaces of genus g is defined as

$$\begin{aligned}&\{(X, f, z) \mid X \text{ closed hyperbolic oriented surface of genus } g,\ z \text{ an } n\text{-tupel of pairwise different}\\ &\text{points in } X \text{ such that the hyperbolic metric has cusps in them,}\\ &f : \Sigma_g \longrightarrow X \text{ orientation preserving diffeomorphism}\}\big/\sim,\end{aligned}$$

where $\sim$ is defined by

$$\begin{aligned}(X, f, z) \sim (Y, g, w) \Longleftrightarrow\, & f \circ g^{-1} \text{ isotopic to a conformal map } \phi : Y \longrightarrow X\\ & \text{with } \phi(w_i) = z_i \text{ for all } i = 1, \ldots, n.\end{aligned}$$

This definition is picturized in Figure 3.1. An analogous definition in terms of almost complex structures on a fixed oriented and closed Riemann surface Σ_g is the following:

Definition 3.13. The Teichmüller space of n-pointed Riemann surfaces of genus g $\mathcal{T}_{g,n}$ is given by

$$\begin{aligned}\{(J, z) \mid J \in \Gamma^\infty(\mathrm{End}(\mathrm{T}\Sigma_g)) \text{ s.t. } J^2 = -\mathrm{id}, J \text{ induces given orientation on } \Sigma_g,\\ z\ n\text{-tupel of pairwise different points in } \Sigma_g\}\big/\mathrm{Diff}_0,\end{aligned}$$

where Diff_0 is as in Definition 3.3 and $\phi \in \mathrm{Diff}_0$ acts via

$$(J, z) \longmapsto (\phi_* J, \phi(z)).$$

3.3 General properties of Teichmüller space

Now that we have seen several definitions of Teichmüller space as point sets we need to talk about structures that exist on this space. First we start by defining a topology. Of course there are again different possibilities due to the different definitions.

Definition 3.14. In the definition $\mathcal{T}_g^{(2)}$ the space inherits the quotient topology from the topology of $\mathcal{A} \subset \Gamma^\infty(\mathrm{End}(\mathrm{T}\Sigma_g))$.

Remark 3.15.

1. One alternative definition of a topology would be to show that $\mathcal{T}_g^{(1)}$ is in bijective correspondance to an open domain of $\mathbb{R}^{6g-6}$, the so-called Fricke space and then define a topology such that the two sets are homeomorphic, see [16].
2. In the definition of $\mathcal{T}_g^{(2)}$ we ignored many technical issues. The space $\mathcal{A}$ is in fact infinite-dimensional and thus we have to be careful which topology we use. In view of the fact that we want to have a differentiable structure on Teichmüller space it would be better to consider everything in the Sobolev $\mathcal{H}^s$ category and then show some regularity results to see that everything works in the C^∞ category as well, see [33] for more details.

Now we pass to a differentiable structure. This can either be achieved by giving explicit coordinates such as the Fenchel–Nielsen coordinates (see Sect. 3.4) and check that the transition functions are smooth or by showing that $\mathcal{T}_g^{(2)}$ inherits a smooth structure as the quotient of the infinite-dimensional manifold $\mathcal{A}$. However, as mentioned in the Remark 3.15 this is technically difficult since the group $\mathcal{D}_0$ with its C^∞-structure is not a Hilbert-Lie-group and thus one has to do a detour via the Sobolev category and then show some regularity result. The best reference for this is probably again [33].

Definition 3.16. The Teichmüller space $\mathcal{T}_g^{(2)}$ has a C^∞-manifold structure coming from the manifold structure of $\mathcal{A}$.

In fact one can even find a natural complex structure on the Teichmüller space of Riemann surfaces of genus g without boundary. It is provided by the Teichmüller theorem or the Bers embedding (see [16]).

Theorem 3.17 (Teichmüller theorem)**.** *The Teichmüller space $\mathcal{T}_g$ can be embedded in $A_2(\Sigma_g)$, which is the space of holomorphic quadratic forms on Σ_g and has dimension $6g-6$.*

Proof. See [16]. □

Remark 3.18.

1. This embedding gives the Teichmüller space a canonical complex structure. As was already mentioned, this complex structure is not canonical anymore if we allow Σ_g to have boundaries.
2. There exist explicit local holomorphic coordinates, called Abresch–Fischer coordinates, see [16].

As we want to consider later a 2-form on Teichmüller space we need to have a good description of its tangent spaces.

Lemma 3.19. *As $\mathcal{T}_g^{(2)} = \mathcal{A} \diagup \mathcal{D}_0$ we formally have*

$$\mathrm{T}_{[J]}\mathcal{T}_g^{(2)} = \{\dot{J} \in \Gamma^\infty(\mathrm{End}(\mathrm{T}\Sigma_g)) \mid J\dot{J} + \dot{J}J = 0\} \diagup$$
$$\{\mathcal{L}_X J \mid X \in \Gamma^\infty(\mathrm{T}\Sigma_g) \text{ such that its flow generates a self-diffeomorphism of } \Sigma_g\}.$$

Here, $\mathcal{L}$ denotes the Lie derivative. As is shown in [33] this is equivalent to considering holomorphic quadratic differentials.

Proposition 3.20. *The space $S_2^{TT}(g_J) := \{h \in S_2(\Sigma_g) \mid \delta_{g_J} h = 0, tr_{g_J} h = 0\}$, where g_J is the metric on Σ_g defined by the complex structure J and the requirement $K(g_J) = -1$, and δ_{g_J} denotes the codifferential of g_J, satisfies*

$$T_{[J]}\mathcal{T}_g \simeq S_2^{TT}(g_J) \simeq A_2((\Sigma_g, J))$$

Proof. See [33]. □

Remark 3.21. Using the embedding of Teichmüller space in the space of quadratic holomorphic forms we can of course deduce directly that its tangent space is given by quadratic holomorphic forms. However, it is also possible to prove this via local calculations. Furthermore this fact can be seen by describing Teichmüller space via Beltrami coefficients, as in [16].

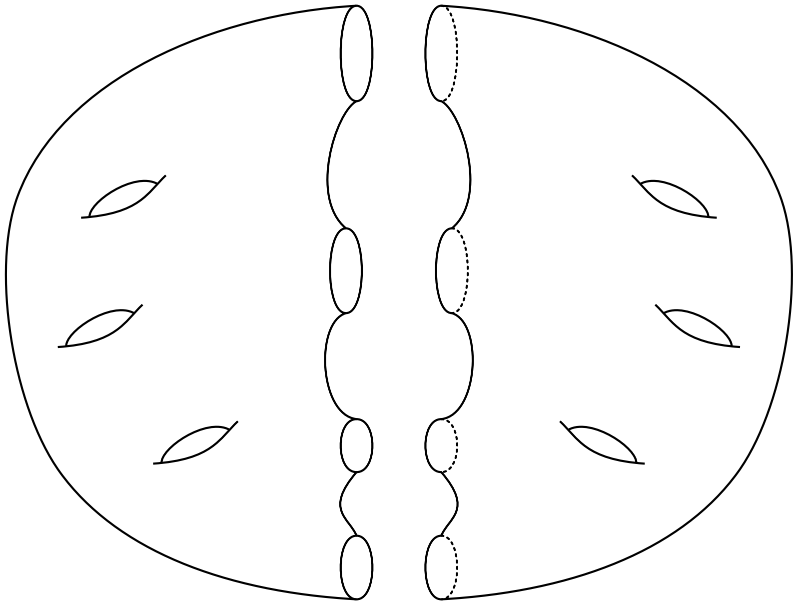


Figure 3.2: The double of a Riemann surface with boundary

It remains to extend these results to surfaces with boundary. The trick of the trade is to double the surface, see Figure 3.2. If the mirror surface has the same complex structure and if the lengths of the boundary curves are fixed we may attach the two equivalent surfaces in a natural way by choosing corresponding geodesics and require that they meet in the same point. In this way we obtain a map $\mathcal{T}_{g,n}(L) \longmapsto \mathcal{T}_{2g+n-1}$. Using Fenchel–Nielsen coordinates (see Sect. 3.4) we choose a system of decomposing curves such that the boundaries along which we doubled the surface are included. Then the image of $\mathcal{T}_{g,n}(L)$ in $\mathcal{T}_{2g+n-1}$ is determined by $l(\beta_i) = L_i, \tau_i = 0$ and that the paramaters with respect to the corresponding curves on the two halfs are equal. Thus it is a submanifold of dimension $(6(2g+n-1)-6-2n)/2 = 6g-6+2n$. See [11], [24] and [17] for more details.
To conclude, we have seen that one can show that the Teichmüller space $\mathcal{T}_g$ is in fact a differentiable manifold of dimension $6g-6$. Its tangent spaces can be given in terms of quadratic holomorphic forms. $\mathcal{T}_g$ is a complex manifold of dimension $3g-3$. The same techniques can be used to show that there exists a topology and a differentiable structure for the Teichmüller space of bordered Riemann surfaces $\mathcal{T}_{g,n}$. It has dimension $6g-6+2n$.

3.4 Fenchel–Nielsen coordinates

In this section we will review the so-called Fenchel–Nielsen coordinates. These are global coordinates on $\mathcal{T}_{g,n}(L)$ which are especially useful for calculations with the Weil–Petersson symplectic form. They come from the following observation. If you consider two hyperbolic pairs of pants, then they are isometric if and only if the lengths of their boundary curves agree. This fact relies on the classification of hyperbolic hexagons in the plane, see Lemma 2.11. Furthermore one can decompose a Riemann surface in pairs of pants such that the total hyperbolic structure is given by the length of the decomposing curves used to obtain the set of pairs of pants and certain twisting parameter which describe how the pairs of pants are glued together. Since the construction is essential for the remaining work we will describe the construction of the coordinates in some detail. This section is based on [18] and [16].
The Fenchel–Nielsen coordinates will be coordinates for $\mathcal{T}_{g,n}(L)$, i.e. $6g-6+2n$ real numbers. We will first show how to construct a hyperbolic surface with genus g and n boundary components from

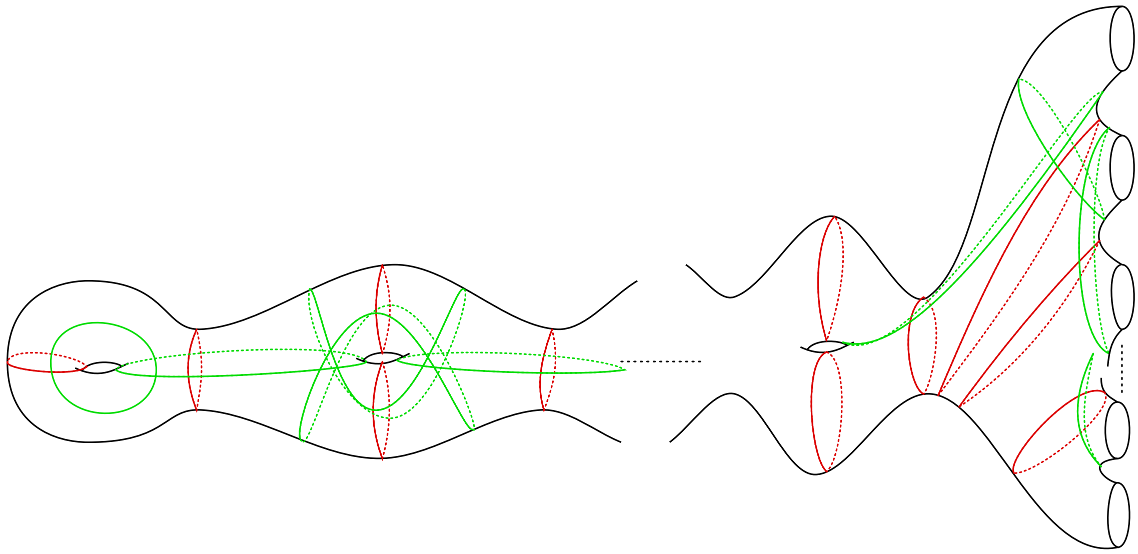


Figure 3.3: The choice of the curves δ_μ (red) and ϵ_μ (green)

the given data $l_1, \dots, l_{3g-3+n} \in \mathbb{R}_+$ and $\theta_1, \dots, \theta_{3g-3+n} \in \mathbb{R}_+$. The Fenchel–Nielsen coordinates depend on a choice of a set of decomposing curves of the surface $\Sigma_{g,n}$.

Notation and definitions

Definition 3.22. A set of decomposing curves $\{\delta_\mu\}_{\mu \in 1,\dots,3g-3+n}$ of $\Sigma_{g,n}$ is a set of $3g-3+n$ closed simple curves of different free homotopy classes which are pairwise disjoint.

Such a set always exists, see [18] and [13]. Now choose $3g-3+n$ closed simple curves $\{\epsilon_\mu\}$ with the property that ϵ_μ intersects δ_λ exactly twice for $\mu = \lambda$ (except for $\mu = \lambda = 1$, then they intersect once)and that they are disjoint for $\mu \neq \lambda$. Since we can consider $\Sigma_{g,n}$ under a homeomorphism we can construct these curves as in Figure 3.3. The set $\{\delta_\mu\}$ decomposes $\Sigma_{g,n}$ in $2g-2+n$ pairs of pants S_ν.

Definition 3.23. Consider some hyperbolic pair of pants S_ν with boundary curves c_1, c_2 and c_3. Then denote by c_i' the unique perpendicular geodesic from c_i to itself. Let c_{ij} be the unique geodesic joining perpendicularly c_i and c_j. Denote the intersection points of c_i' with c_i by w_i and w_i' and those of c_{ij} with c_i and c_j by z_i and z_j', respectively. Orientation and nomenclature issues are summarized in Figure 3.4.

We will keep the notation and add an index ν if the objects are those in the pair of pants S_ν. Since the curves c_{ij} and c_i' are uniquely determined by the lengths of the boundary curves of the hyperbolic pair of pants one can see that they depend continously on the boundary lengths. Now we will construct a point in the Teichmüller space $\mathcal{T}_{g,n}(L)$ without any twisting. This construction will allow us afterwards to implement the twisting coordinates θ and by comparison with this surface to determine the twisting coordinates of a given point in Teichmüller space.

Construction of (X_0, f_0)

Having the decomposition of the model Riemann surface $\Sigma_{g,n}$ in mind we take $2g-2+n$ pairs of pants with the lengths given by the $3g-3+n$ real l values and the n parameters from L. Now we glue together the pairs of pants along the joined boundaries as prescribed by the decomposition of $\Sigma_{g,n}$. There are two cases, either the boundaries glued together belong to the same pair of pants or to two different ones.

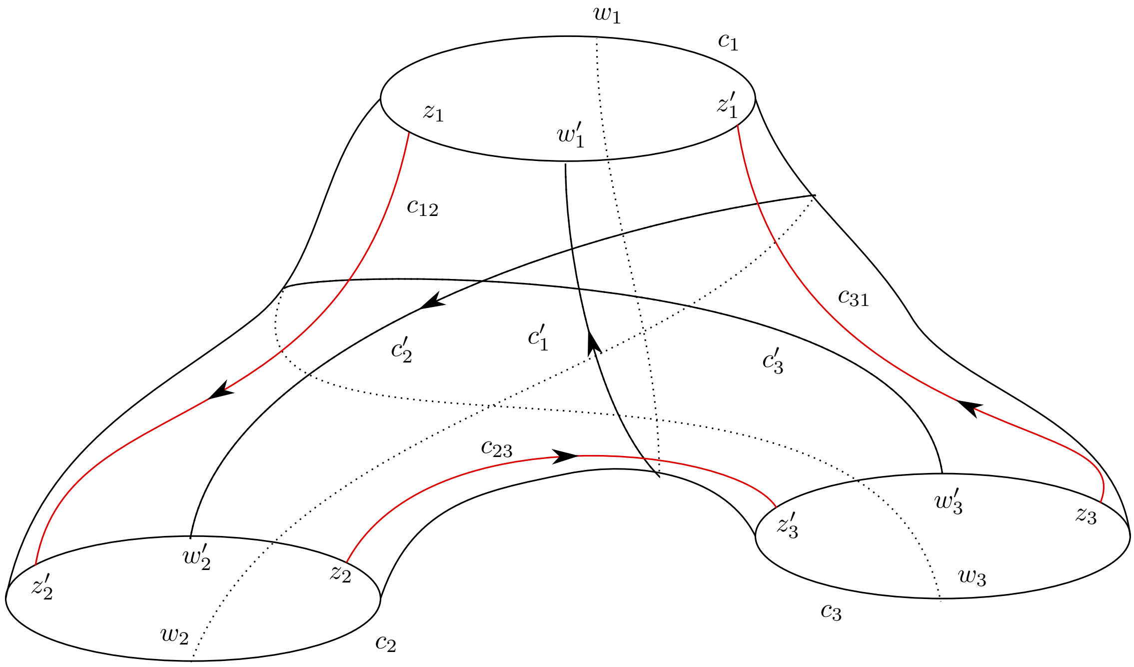


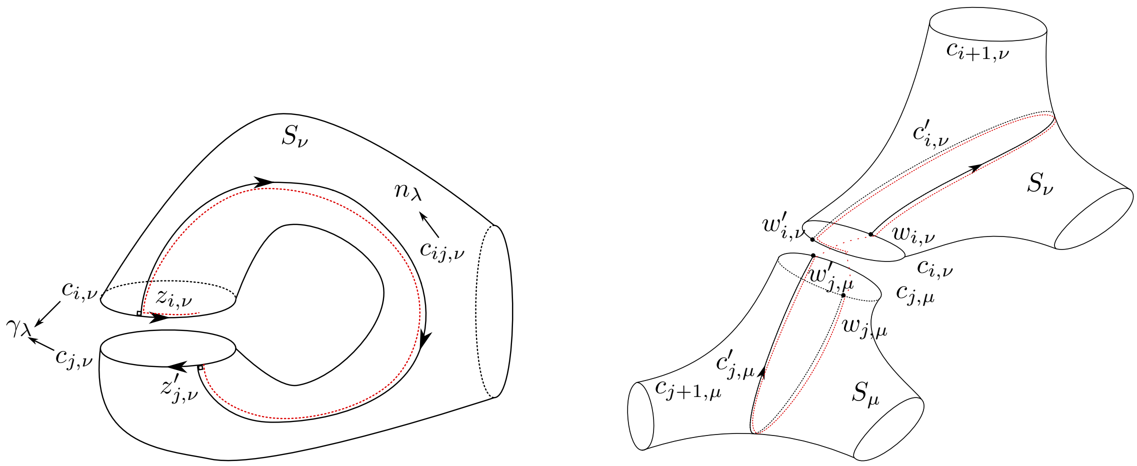
Figure 3.4: Names and orientation of the geodesics on a pair of pants necessary to fix the choices for the Fenchel–Nielsen coordinates. Note that the definitions are cyclic modulo 3

In the first case, suppose we glue together $c_{i\nu}$ and $c_{j\nu}$ of S_ν. Then we identify the two boundary curves such that $z_{i\nu}$ becomes identified with $z'_{j\nu}$ and we require for the marking $f_0 : \Sigma_{g,n} \longrightarrow X$ that $f_0(\epsilon_\lambda) = n_\lambda$ if we call the curve $c_{ij\nu}$ on the glued surface n_λ, where λ is determined by the comparison of the free homotopy type of ϵ_λ and the image of $c_{ij\nu}$ on the surface. Furthermore we call the image of $c_{i\nu}$ and $c_{j\nu}$ on the glued surface γ_λ. Although this seems pretty complicated the idea is actually quite intuitive, see Figure 3.5a.
Now suppose we need to glue together S_ν and S_μ with $\nu < \mu$ and call the two curves which are identified $c_{i\nu} \sim c_{j\mu}$ and their image on the glued surface γ_λ. We glue the two pairs of pants together in such a way that $w_{i\nu}$ and $w'_{j\mu}$ are identified. In order to fix the marking f_0 we require that $f_0(\epsilon_\lambda)$ is homotopic to $n_\lambda = c'_{j\mu} * \overline{c_{i\nu}} * c'_{i\nu}$, where $\overline{c_{i\nu}}$ is the piece of $c_{i\nu}$ from $w'_{i\nu}$ to $w_{j\mu}$, see Figure 3.5b.
All in all we have obtained a hyperbolic surface X_0 with genus g and n boundary components. The marking $f_0 : \Sigma_{g,n} \longrightarrow X_0$ is defined by the requirement that it is a homeomorphism and by the $3g - 3 + n$ conditions on the free homotopy classes of ϵ_μ. This fixes f_0 up to homotopy and thus $(X_0, f_0) \in \mathcal{T}_{g,n}(L)$.

Construction of $(l, \theta) \longmapsto (X, f)$

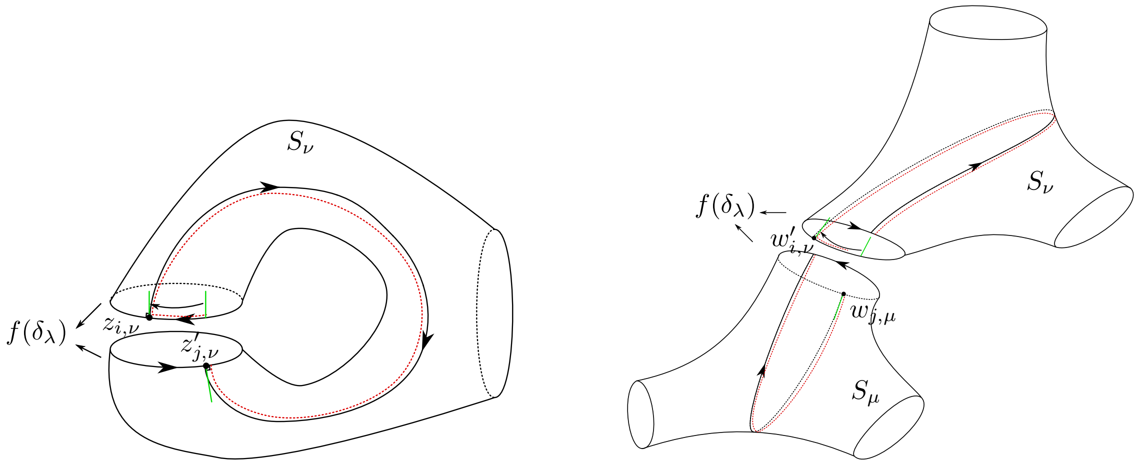
Now suppose we want to construct a surface which is twisted along the gluing curves. Thus let $l_1, \ldots, l_{3g-3+n} \in \mathbb{R}_+$ and $\theta_1, \ldots, \theta_{3g-3+n} \in \mathbb{R}$ be given. First construct (X_0, f_0) as before and parametrize the curves $\gamma_\lambda \sim f_0(\epsilon_\lambda)$ proportional to arclength measured with respect to the hyperbolic structure on X. Then for all $\lambda \in 1, \ldots, 3g - 3 + n$ we cut X_0 along γ_λ to obtain two copies $c_{i\nu}$ and $c_{j\mu}$ of γ_λ. Then we rotate the one with the higher index (i.e. $(i,\nu) > (j,\mu) \iff \nu > \mu$ or $\mu = \nu, i > j$) against the other by θ_λ and glue the surfaces back together. Here gluing together means that we change to a homeomorphic surface with a different hyperbolic structure and a different marking. The marking is changed in the obvious way, see Figure 3.6.
Thus we have constructed a map

$$\mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n} \longrightarrow \mathcal{T}_{g,n}(L).$$



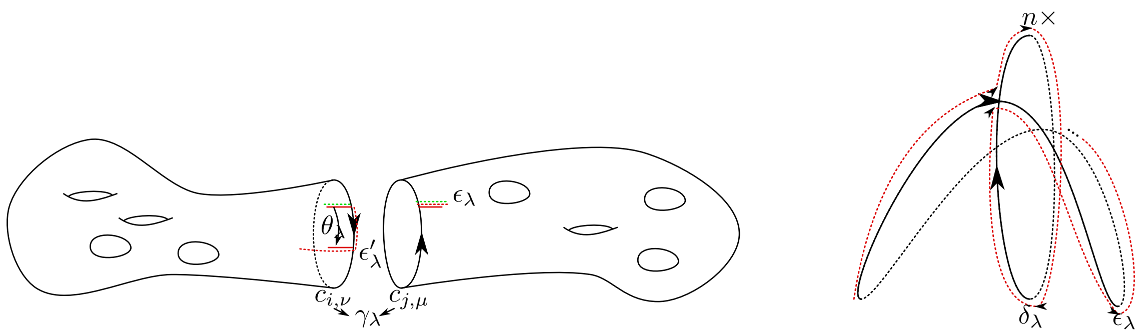
(a) (b)

Figure 3.5: The gluing of the corresponding curves



(a) (b)

Figure 3.6: The twisting procedure for changing the hyperbolic structure and the marking. Note that the picture is to be interpreted differently compared to Figures 3.5a and 3.5b. Here we cut an already given pair of pants along the curve and reglue it in a different way



(a) The determination of θ^0_λ by comparison of the corresponding points on two hyperbolic pairs of pants of given length with the actual endpoints of the given ϵ_λ

(b) The Dehn twist around δ_λ. Note that one needs to go around δ_λ n-times at the second intersection, too

Figure 3.7

Construction of $(X, f) \longmapsto (l, \theta)$

Having fixed a set δ and ϵ of curves on $\Sigma_{g,n}$ as before we can first of all determine the hyperbolic lengths of $f(\delta_\mu)$, i.e. $l_\mu = l_{\text{hyp}}(f(\delta_\mu))$. This already gives us half the coordinates. Furthermore we get the $2g-2+n$ pairs of pants as well as the distinguished points $w_{i\nu}, w'_{i\nu}, z_{i\nu}, z'_{i\nu}$. Let us define

$$\theta^0_\lambda = \text{oriented angle between } \begin{cases} z_{i\mu} \text{ and } z'_{j\mu} \text{ for } \mu = \nu \\ w_{i\mu} \text{ and } w'_{j\nu} \text{ for } \mu \neq \nu \end{cases},$$

see Figure 3.7a.
Now we construct $(X', f') \in \mathcal{T}_{g,n}(L)$ as before with the parameters $l_1, \dots, l_{3g-3+n}$ and $\theta^0_1, \dots, \theta^0_{3g-3+n}$. We have now the curves $n_\lambda, \gamma_\lambda$ on X' and δ_λ and ϵ_λ on $\Sigma_{g,n}$. The marking f' satisfies by construction $f'(\delta_\lambda) \sim \gamma_\lambda$. However, it is still possible that $f'(\epsilon_\lambda) \nsim n_\lambda$ because only the homotopy classes of $n_{\lambda_{|S_\nu}}$ are fixed. Thus the two curves could still differ by a power of γ_λ, i.e. we need to determine $n \in \mathbb{Z}$ such that $n_\lambda \sim f'(\epsilon_\lambda \delta^n_\lambda)$. Here $\epsilon_\lambda \delta^n_\lambda$ means the curve obtained by cutting ϵ_λ at the intersection points with δ_λ and moving around δ_λ in its positive direction n-times before continuing along ϵ_λ, see Figure 3.7b. Then we define $\theta_\lambda = \theta^0_\lambda + 2\pi n$.
This defines the global Fenchel–Nielsen coordinates of the Teichmüller space $\mathcal{T}_{g,n}(L)$.

Remark 3.24. 1. As was already pointed out we obtain a homeomorphism

$$\mathcal{T}_{g,n}(L) \longrightarrow \mathbb{R}^{3g-3+n}_+ \times \mathbb{R}^{3g-3+n}.$$

2. This construction works also in the case that $L = 0$, i.e. in the case of punctured Riemann surfaces, because hyperbolic pentagons with one ideal vertex are uniquely determined up to isometries by the lengths of two alternating edges.

3. This definition of Fenchel–Nielsen coordinates depended on several choices. For different choices one obtains real analytic transition maps, i.e. we found a real analytic structure of Teichmüller space, in particular a smooth structure and a topology. In fact it coincides with the already defined manifold structure, see [18] and [1].

4. We saw that the operation of cutting a surface along a closed simple non-peripheral hyperbolic geodesic, turning the two cut surfaces against each other and then regluing played a special role. This operation is called a twist $\text{tw}^t_\gamma : \mathcal{T}_{g,n}(L) \longrightarrow \mathcal{T}_{g,n}(L)$ and will play a very important role later on. Here, t denotes how far the rotation is done. The appearence of the summand n at the last step has to do with the possibility of applying total twists of length l_γ around γ. These twists are called Dehn twists and are generators of the mapping class group,

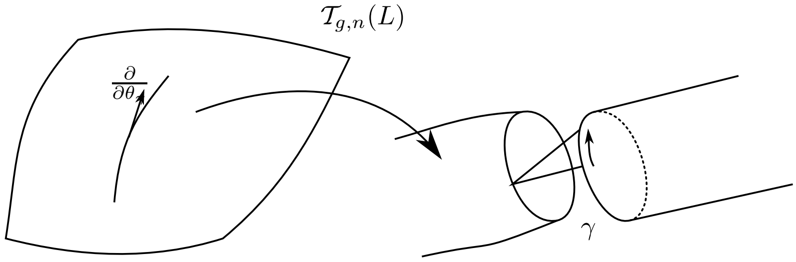


Figure 3.8: The Fenchel–Nielsen twist

a set which will play a crucial role when investigating the relation between Teichmüller and moduli spaces, see Figure 3.8.

5. By choosing Fenchel–Nielsen coordinates with respect to a set of decomposing curves containing a given curve γ we see that the coordinate vector field $\frac{\partial}{\partial\theta_\gamma}$ has the flow

$$(l_1, \dots, l_\gamma, \dots, l_{3g-3+n}, \theta_1, \dots, \theta_\gamma + t, \dots, \theta_{3g-3+n})$$

which shows that the twist tw_γ^t is in fact the flow of $\frac{\partial}{\partial\theta_\gamma}$, see Figure 3.8.

3.5 Weil–Petersson symplectic structure

In this section we will introduce a geometric structure on the Teichmüller space $\mathcal{T}_{g,n}(L)$. Since it is a differentiable manifold we can in principal do Riemannian geometry or symplectic geometry on the Teichmüller space and investigate its structure a bit further. In fact there exist many different metrics on $\mathcal{T}_{g,n}(L)$, see [22]. Although we defined coordinates in the last chapter by looking at points in $\mathcal{T}_{g,n}(L)$ as marked Riemann surfaces, we will now define the so-called Weil–Petersson symplectic structure in terms of almost complex structures on $\Sigma_{g,n}$. Recall from Proposition 3.20 that $\mathrm{T}_{[J]}\mathcal{T}_{g,n}(L) \simeq A_2(\Sigma_{g,n}, J)$.

Definition 3.25. Let $\xi \mathrm{d}z^2$ and $\gamma \mathrm{d}z^2$ be two holomorphic quadratic differentials on $(\Sigma_{g,n}, J)$, i.e. tangent vectors in $\mathrm{T}_{[J]}\mathcal{T}_{g,n}(L)$. Using complex coordinates $z = x + \mathrm{i}y$ such that $g(J)_{ij} = \lambda\delta_{ij}$ we define

$$\langle \xi \mathrm{d}z^2, \gamma \mathrm{d}z^2 \rangle_{\mathrm{WP},[J]} := \mathrm{Re} \int_{\Sigma_{g,n}} \frac{\xi\overline{\gamma}}{\lambda} \mathrm{d}x\mathrm{d}y,$$

the so-called Weil–Petersson metric.

For more details, like well-definedness and certain properties, see [33] and [39]. It is in fact related to the L^2-metric on the space of almost-complex structures $\mathcal{A}$. For $X, Y \in \mathrm{T}_{[J]}\mathcal{T}_{g,n}(L)$ we can define

$$\langle X, Y \rangle_{L^2,[J]} := \int_{\Sigma_{g,n}} \mathrm{tr}(\tilde{X}\tilde{Y}) \mathrm{dvol}_{g(J)}, \tag{3.1}$$

where $\tilde{X}$ and $\tilde{Y}$ are horizontal lifts of X and Y to $\mathcal{A}$ with respect to the metric on the right hand side of (3.1) and thus they are global sections of $\mathrm{End}(\Sigma_{g,n})$. Then we have

Lemma 3.26. *For the two metrics defined on $\mathcal{T}_{g,n}(L)$ we have*

$$\langle \cdot, \cdot \rangle_{WP} = \frac{1}{2} \langle \cdot, \cdot \rangle_{L^2}.$$

Proof. By local calculation, see [33]. □

The Weil–Petersson metric has several interesting properties.

Theorem 3.27. *1. The Weil–Petersson metric is not complete.*

2. It is Kähler.

3. It is invariant under the mapping class group $Mod_{g,n}$.

4. Its sectional curvature is negative and its Ricci curvature is negative and bounded uniformly away from zero.

Proof. See [33] and [39]. □

For us the most important detail is that it is Kähler because this means that the Teichmüller space is also a symplectic manifold with the Weil–Petersson symplectic form, given by the following.

Definition 3.28. The Weil–Petersson symplectic form on $\mathcal{T}_{g,n}(L)$ is given in complex coordinates by

$$\omega_{\mathrm{WP}}(\xi \mathrm{d}z^2, \gamma \mathrm{d}z^2) = -\mathrm{Im} \int_{\Sigma_{g,n}} \frac{\xi \overline{\gamma}}{\lambda} \mathrm{d}x \mathrm{d}y. \tag{3.2}$$

Remark 3.29. 1. We have a $\mathrm{Mod}_{g,n}$ invariant symplectic form on $\mathcal{T}_{g,n}(L)$. In the next chapter we will see that the moduli space of Riemann surfaces satisfies $\mathcal{M}_{g,n}(L) \simeq \mathcal{T}_{g,n}(L) \big/ \mathrm{Mod}_{g,n}$. Thus this metric descends to the moduli space in the orbifold sense.

2. In some sense the Weil–Petersson metric or symplectic form reflects the local perturbation behaviour of the hyperbolic structure. Since it is defined in terms of the hyperbolic metric on the surface of the corresponding point and because the holomorphic quadratic differentials can be interpreted as perturbations of the hyperbolic or complex structure this seems reasonable. A calculation making explicit this argument can be found in [38].

One more fact about the Weil–Petersson symplectic form will be crucial to our further calculations, namely Wolpert's formula.

Theorem 3.30 (Wolpert)**.** *In Fenchel–Nielsen coordinates with respect to some set of decomposing curves $\{c_i\}_{i=1}^{3g-3+n}$ of $\Sigma_{g,n}$ with length function l_i and twisting functions τ_i the Weil–Petersson symplectic form is given by*

$$\omega_{\mathrm{WP}} = \sum_{i=1}^{3g-3+n} \mathrm{d}l_i \wedge \mathrm{d}\tau_i.$$

Proof. See [33], [16] and [39]. □

3.6 Connection to moduli spaces

Although we have been intensively talking about Teichmüller spaces one should remark that the maybe more fundamental space is the moduli space of Riemann surfaces. In fact, when classifying complex Riemann surfaces with given topological type, i.e. of given genus g and n boundary components one is interested in the space of all complex Riemann surfaces modulo a natural equivalence, in this case biholomorphicity. Since the homeomorphism class of a Riemann surface is determined by g and n and since in dimension two homeomorphic manifolds are diffeomorphic this is a task that comes very quickly to our minds. It means we are interested in the space

$$\{(X, J) \mid X \text{ a Riemann surface of genus } g \text{ with } n \text{ boundary components}, \\ J \text{ an almost complex structure on } X\} \big/ \sim,$$

where $(X, J) \sim (Y, J') \Longleftrightarrow \exists f : X \longrightarrow Y$ biholomorphism with respect to J and J'. However, similiar to Teichmüller space there are various viewpoints on how one can define this space, therefore we will state two of them.

Definition 3.31. Let $\Sigma_{g,n}$ be a Riemann surface of genus g and with n boundary components. Then the moduli space of Riemann surfaces of genus g and n boundary components of length L is defined as

1. $\mathcal{M}_{g,n}(L) = \{(X, J) \mid X$ a Riemann surface of genus g and n boundary components, J an almost complex structure on $X\} / \sim$, where, as above $(X, J) \sim (Y, J') \Longleftrightarrow \exists f : X \longrightarrow Y$ biholomorphism
2. $\mathcal{M}_{g,n}(L) = \{X$ a hyperbolic Riemann surface with geodesic boundary $\beta_1, \ldots, \beta_n$ of lengths $L_1, \ldots, L_n\} / \sim$, where $X \sim Y \Longleftrightarrow \exists \phi : X \longrightarrow Y$ an isometry mapping $\beta_i \subset X$ on $\beta_i \subset Y$

So far we have defined the moduli space as a point set only, however, in order to do geometry on it we need to give it a topology and, if possible, a smooth structure or something similar. In fact, the definition is very close to two definitions of the Teichmüller space.

Lemma 3.32. *The relation between the Teichmüller space and the moduli space as sets is the following*

$$\mathcal{M}_{g,n}(L) = \mathcal{T}_{g,n}(L) / Mod_{g,n}.$$

Proof. See [16]. □

Thus we obtain a topology on the moduli space by taking the quotient topology. This coincides with other possible definitions. However, $\mathrm{Mod}_{g,n}$ does not act freely on $\mathcal{T}_{g,n}(L)$ such that we do not obtain a smooth manifold structure on $\mathcal{M}_{g,n}(L)$. However, since the automorphism group of a Riemann surface is finite we obtain the following statement.

Proposition 3.33. *The moduli space $\mathcal{M}_{g,n}(L)$ is an orbifold, i.e. it is locally the quotient of a differentiable manifold by a finite group.*

Proof. See [13] and [30]. □

Although this means that $\mathcal{M}_{g,n}(L)$ is not as beautiful as one might hope we can nevertheless do many things as in the case of manifolds. Recall that an orbifold might have singularities of measure zero, i.e. we can integrate over orbifolds as over manifolds, we just need to consider the smooth part. Furthermore away from the singularities we have well-defined tangent spaces and can talk about forms and vectorfields. Of course the construction of bundles over orbifolds is possible. There is just a slight change of the meaning of these objects. However, as this causes only technical diffiulties which can be resolved by standard techniques, see e.g. [25], we will not care about these issues and just act as if the moduli space was a manifold. We just need to remember that in all theorems about the moduli space we have to add "in the sense of orbifolds" to the statements.

4 Weil–Petersson-volumes of moduli spaces

4.1 Integration over the moduli space

4.1.1 Definitions

In this chapter we will use symplectic reduction and the Weil–Petersson geometry of the moduli space of Riemann surfaces in order to give a formula for the integral of certain kinds of functions over the moduli space. However, in order to be precise we will need to introduce a lot of notation, which will need some time at the beginning of this section.
First, we define a multicurve to be a formal finite non-negative linear combination of good curves, i.e. $\gamma = \sum_{i=1}^{k} c_i\gamma_i$, where γ_i is a good curve and the c_i's are the real non-negative coefficients. Recall that a good curve is an essential, non-peripheral, closed, simple curve and we require that they are all in different free homotopy classes and disjoint. If γ is a multicurve on the surface X then we denote by Γ the k-tupel of curves, i.e. $\Gamma = (\gamma_1, \dots, \gamma_k) \subset X^k$. Now we extend the defintion of the length function on the moduli space to multicurves as follows.

Definition 4.1. For $(X, f) \in \mathcal{T}_{g,n}(L)$ and γ a multicurve on the Riemann surface $\Sigma_{g,n}$ we define

$$l_\gamma(X) = \sum_{i=1}^{k} c_i l_{\gamma_i}(X),$$

where $l_{\gamma_i}(X)$ denotes the hyperbolic length of the geodesic representative in the free homotopy class of $f(\gamma_i) \subset X$.

Furthermore we can use the length function to assign to a function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ an induced function on the moduli space.

Definition 4.2. For $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, γ a multicurve on $\Sigma_{g,n}$ and $[X] \in \mathcal{M}_{g,n}(L)$ we define

$$\begin{aligned} f_\gamma : \mathcal{M}_{g,n}(L) &\longrightarrow \mathbb{R}_+ \\ [X] &\longmapsto \sum_{\phi \in \mathrm{Diff}_+(\Sigma_{g,n}, X)/\sim} f(l_{\phi(\gamma)}(X)), \end{aligned}$$

where $\mathrm{Diff}_+(\Sigma_{g,n}, X)/\sim$ denotes the set of all diffeomorphisms from $\Sigma_{g,n}$ to X modulo isotopy. Here, $l_{\phi(\gamma)}(X)$ denotes the length of the multicurve $\phi(\gamma)$ on the hyperbolic surface X.

Remark 4.3. The last definition makes sense because we sum over all diffeomorphisms and thus the value is independent of the chosen representative. However, as we know, the mapping class group acts transitively on the set $\mathrm{Diff}_+(\Sigma_{g,n}, X)$ by $\phi \longmapsto \phi \circ g^{-1}$ for $g \in \mathrm{Mod}_{g,n}$. Thus we can rewrite the last definition as

$$f_\gamma(X) = \sum_{g \in \mathrm{Mod}_{g,n}} f(l_{g\cdot\phi(\gamma)}(X)),$$

where $\phi : \Sigma_{g,n} \longrightarrow X$ is any diffeomorphism and $g \cdot \phi = \phi \circ g^{-1}$.

It is those kind of functions which we will integrate over the moduli space later on. We will also need to talk about symmetries of mutlicurves.

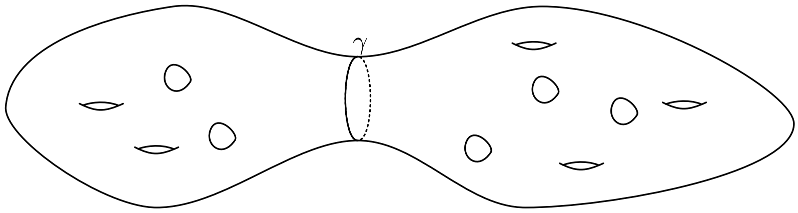


Figure 4.1: A good curve γ

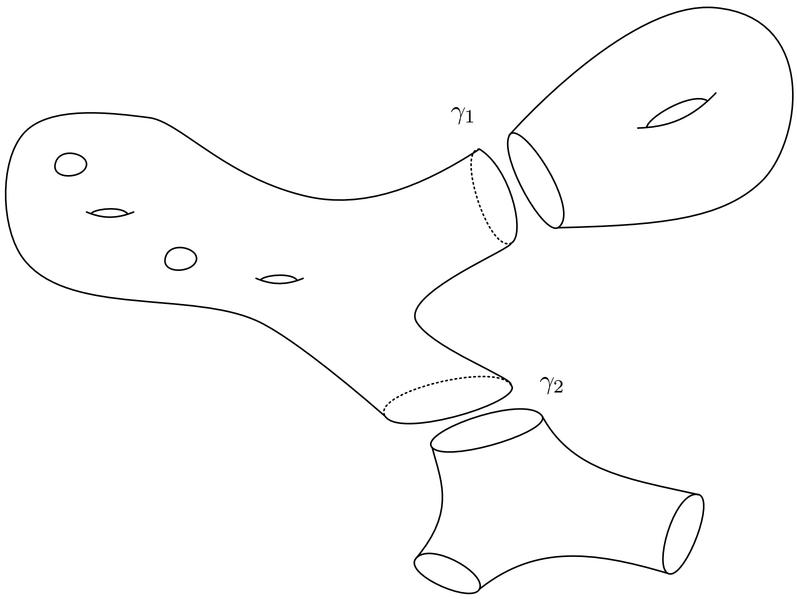


Figure 4.2: A Riemann surface cutted along a multicurve $\gamma = \gamma_1 + \gamma_2$

Definition 4.4. Let γ be a multicurve on $\Sigma_{g,n}$. Then we define

1. $\mathrm{Stab}(\gamma) := \{g \in \mathrm{Mod}_{g,n} \mid g \cdot \gamma = \gamma$ up to free homotopy$\}$
2. $\mathrm{Sym}(\gamma) := \mathrm{Stab}(\gamma) \diagup \bigcap_{i=1}^{k} \mathrm{Stab}(\gamma_i)$.

Example 4.5. Suppose $\gamma = \gamma_1 + \gamma_2$, where γ_1 and γ_2 are good curves. Then the self-homeomorphism must either preserve γ_1 and γ_2 or interchange them. Thus,

$$\mathrm{Stab}(\gamma) = \{h \in \mathrm{Stab}(\gamma_1) \cap \mathrm{Stab}(\gamma_2)\} \cup \{h \in \mathrm{Mod}_{g,n} \mid h(\gamma_1) = \gamma_2 \text{ and } h(\gamma_2) = \gamma_1\}.$$

Thus we have $|\mathrm{Sym}(\gamma)| \in \{1, 2\}$ depending on wheter there exists an element interchanging the two geodesics or not. Now call $\Sigma_{g,n}(\gamma)$ the Riemann surface obtained by cutting $\Sigma_{g,n}$ along the curves γ_i, see Figure 4.2. Suppose there is $h \in \mathrm{Mod}_{g,n}$ such that $h(\gamma_1) = \gamma_2$ and vice verca. Then, $\Sigma_{g,n}(\gamma_1) \simeq \Sigma_{g,n}(\gamma_2)$ by Lemma 3.7. Therefore, if $|\mathrm{Sym}(\gamma_1 + \gamma_2)| = 2$ then $\Sigma_{g,n}(\gamma_1)$ is homeomorphic to $\Sigma_{g,n}(\gamma_2)$.
Now suppose we have that $\Sigma_{g,n}(\gamma_1) \simeq \Sigma_{g,n}(\gamma_2)$. Then by Lemma 3.7 there exists an $h \in \mathrm{Mod}_{g,n}$ such that $h(\gamma_1) = \gamma_2$. If $h(\gamma_2) = \gamma_1$ we see that there exists an element in the mapping class groups which stabilizes the multicurve $\gamma_1 + \gamma_2$ but which is not in the stabilizer group of the individual curves. Thus $|\mathrm{Sym}(\gamma)| = 2$. If $h(\gamma_2) \neq \gamma_1$, we notice that $\Sigma_{g,n}(\gamma_1) \simeq \Sigma_{g,n}(\gamma_2) \simeq \Sigma_{g,n}(h(\gamma_2))$ and thus there exists $g \in \mathrm{Stab}(\gamma_2)$ such that $g^{-1}(\gamma_1) = h(\gamma_2)$. Therefore $g \circ h$ interchanges γ_1 and γ_2 and we have again $|\mathrm{Sym}(\gamma)| = 2$.

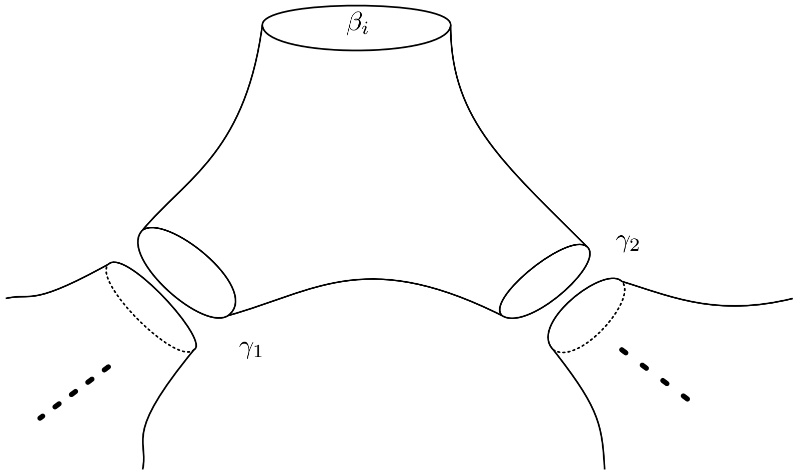
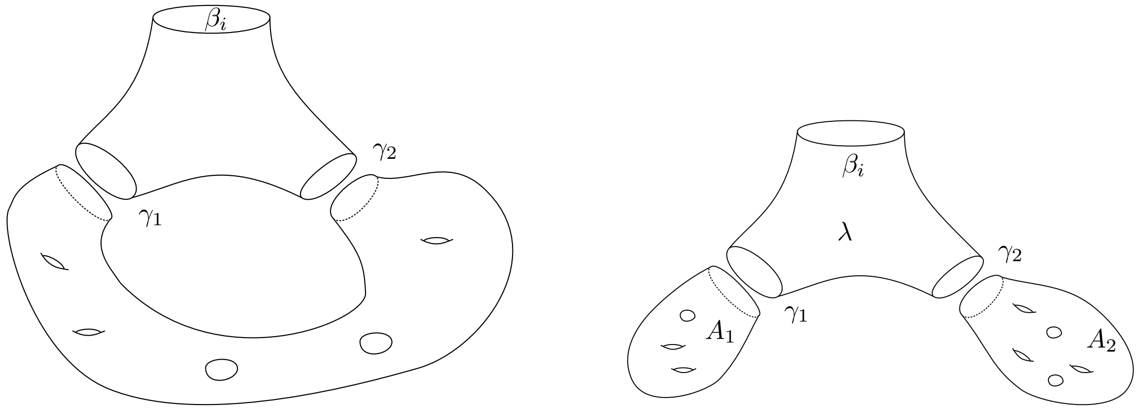


Figure 4.3: A pair of pants bounded by a multicurve and a boundary component



(a) (b)

Figure 4.4: The two possibilities for the cut surface

Now let us restrict to the case we will actually need later, i.e. the two curves γ_1 and γ_2 bound an embedded pair of pants Λ in $\Sigma_{g,n}$ where the third boundary is given by a boundary β_i of $\Sigma_{g,n}$, see Figure 4.3. There are two possibilities, either $\Sigma_{g,n} \setminus \Lambda$ is connected or disconnected. If it is connected we know that $|\mathrm{Sym}(\gamma)| = 2$ by the last lines, since in this case the two cut surfaces are homeomorphic, see Figure 4.4a. If it is not connected let us call the two connected components A_1 and A_2, see Figure 4.4b. Then either A_1 and A_2 are homeomorphic by a homeomorphism fixing the boundary components, in which case $|\mathrm{Sym}(\gamma)| = 2$, or they are not, in which case $|\mathrm{Sym}(\gamma)| = 1$. Since A_1 and A_2 are disjoint, the first case can only happen if the genera of A_1 and A_2 agree and they have no boundaries. Thus we have shown:

Lemma 4.6. *For $\gamma = \gamma_1 + \gamma_2$ such that γ_1 and γ_2 together with a boundary component of $\Sigma_{g,n}$ bound an embedded pair of pants Λ we have $|Sym(\gamma)| = 2$ if and only if either*

1. *$\Sigma_{g,n} \setminus \Lambda$ is connected, or*

2. *$\Sigma_{g,n} \setminus \Lambda \simeq \Sigma_{g',1} \dot{\cup} \Sigma_{g',1}$.*

In all other cases we have $|Sym(\gamma)| = 1$.

The explicit integration will be done on a certain covering of the moduli space instead of on the moduli space itself. Thus we need a formula expressing the integrals of functions of the above kind on covering spaces. Let $\pi : E \longrightarrow B$ be a covering map between manifolds and suppose that we have a volume form v on B. Then we can pull back the volume form to E. Furthermore we can in principle push forward a function f on E by summing over its values on the preimage of a point under π. However, since the covering is infinite this may not be well defined. We have the following:

Lemma 4.7. *Let $\pi : E \longrightarrow B$ be a covering map and v a volume form on B. Furthermore, let f be a non-negative function in $L^1(E, \pi^* v)$ and define $\pi_* f$ as a function $B \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ by*

$$\pi_* f(x) = \sum_{y \in \pi^{-1}(x)} f(y).$$

Then $\pi_* f \in L^1(B, v)$ and

$$\int_B \pi_* f \, dv = \int_E f \, d\pi^* v.$$

Proof. Pick a family of charts on B and a partition of unity subordinate to these charts and call them $\{\alpha_k, U_k\}$. Then we have

$$\begin{aligned} \int_B \pi_* f \mathrm{d}v &= \sum_k \int_{U_k} \alpha_k \pi_* f \mathrm{d}v \\ &= \sum_k \int_{U_k} \sum_{y \in \pi^{-1}(x)} \alpha_k(x) f(y) \mathrm{d}v(x) \\ &= \sum_k \int_{\pi^{-1}(U_k)} \pi^* \alpha_k(y) f(y) \mathrm{d}\pi^* v(y) \\ &= \int_E f \pi^* v < \infty. \end{aligned}$$

□

Now we want to define one of the most important spaces we will need later. It will be a certain covering of the moduli space over which we will integrate by virtue of the last lemma. Let Γ be a k-tupel of good curves on $\Sigma_{g,n}$. Then, any $h \in \mathrm{Mod}_{g,n}$ acts on Γ by acting on the curves, i.e.

$$h \cdot \Gamma = (h \cdot \gamma_1, \ldots, h \cdot \gamma_k). \tag{4.1}$$

Definition 4.8. Let Γ be a k-tupel of good curves on $\Sigma_{g,n}$ and $[X] \in \mathcal{M}_{g,n}(L)$ the equivalence class of a hyperbolic surface X. Then $\mathcal{O}_\Gamma^X$ is defined to be the image of the $\mathrm{Mod}_{g,n}$-orbit of free homotopy classes of Γ on X, i.e.

$$\begin{aligned} \mathcal{O}_\Gamma^X &= \{([\phi(\gamma_1)], \ldots, [\phi(\gamma_k)]) \mid \phi \in \mathrm{Diff}_+(\Sigma_{g,n}, X) / \sim\} \\ &= \{([\phi(h \cdot \gamma_1)], \ldots, [\phi(h \cdot \gamma_k)]) \mid h \in \mathrm{Mod}_{g,n}\}, \end{aligned}$$

where in the last line ϕ is any diffeomorphism $\phi : \Sigma_{g,n} \longrightarrow X$, see Figure 4.5.

Since ϕ ís arbitrary it is not possible to assign to a homotopy class $[\gamma]$ on $\Sigma_{g,n}$ a unique homotopy class on X. However, the set $\mathcal{O}_\gamma^X$ is the natural assignment because it includes all possible homotopy classes of $[\gamma]$ on X. Now we define the following space:

Definition 4.9. Let $L \in \mathbb{R}_+^n$ and Γ be a k-tupel of curves as before. Then

$$\mathcal{M}_{g,n}(L)^\Gamma = \{(X, \eta) \mid [X] \in \mathcal{M}_{g,n}(L), \eta \in \mathcal{O}_\Gamma^X\} / \sim,$$

where the equivalence relation is defined as follows

$$(X, \eta) \sim (Y, \tau) \Longleftrightarrow \exists h : X \longrightarrow Y \text{ or. preserving isometry s.t. } h(\eta) = \tau.$$

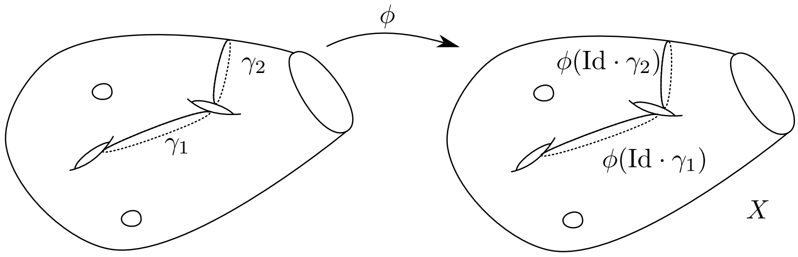


Figure 4.5: The set $\mathcal{O}_\Gamma^X$ denotes the set of all such mutlicurves when replacing $\phi \circ \mathrm{Id}$ by any diffeomorphism $\Sigma_{g,n} \longrightarrow X$

Here h acts on η by taking the free homotopy class of a representative of the free homotopy class for each entry in the tupel, i.e. $h(\eta)$ is defined by $h_*(\eta_i)$, if η_i is the i-th free homotopy class.

Now we want to show the following

Lemma 4.10. *Let all the spaces be defined as before. Then*

$$\mathcal{M}_{g,n}(L)^\Gamma \simeq \mathcal{T}_{g,n}(L) \diagup G_\Gamma,$$

where $G_\Gamma = \bigcap_{i=1}^k \mathrm{Stab}(\gamma_i) \subset \mathrm{Mod}_{g,n}$ *and* $\simeq$ *means diffeomorphic.*

Proof. Define a mapping $\rho : \mathcal{T}_{g,n}(L) \diagup G_\Gamma \longrightarrow \mathcal{M}_{g,n}(L)^\Gamma$ by

$$[(X, f)] \longmapsto \big[X, ([f(\gamma_1)], \ldots, [f(\gamma_k)])\big], \tag{4.2}$$

where $[\ldots]$ means the equivalence class with respect to the corresponding equivalence which is clear from the context. We will show that ρ is well defined, bijective and smooth in the sense of orbifolds.

Well-definedness We need to show

$$(Y, g) \sim_{G_\Gamma} (X, f) \Longrightarrow \exists h : X \longrightarrow Y \text{ or. preserving isometry s.t. } h_*([f(\gamma_i)]) = g_*([\gamma_i]). \tag{4.3}$$

By assumption there exists $l \in G_\Gamma$ such that $(X, f) \sim_{\mathcal{T}_{\}, \backslash}(\mathcal{L})} (Y, g \circ l^{-1})$, i.e. $f \circ l \circ g^{-1}$ is homotopic to an orientation preserving isometry $\phi : Y \longrightarrow X$. Thus $h := \phi^{-1} : X \longrightarrow Y$ is an orientation preserving isometry homotopic to $g \circ l^{-1} \circ f^{-1}$. Thus, they induce the same mapping on the free homotopy classes. Therefore,

$$h_*([f(\gamma_i)]) = g_* l_*^{-1} f_*^{-1}([f(\gamma_i)]) = g_* l_*^{-1}([\gamma_i]) = g_*([\gamma_i]),$$

because $l \in \bigcap_{i=1}^k \gamma_i$ and thus its induced map fixes the homtopy class of all γ_i. Therefore the map ρ is well defined.

Injectivity Assume we have two elements in $\mathcal{M}_{g,n}^\Gamma(L)$ such that $\rho([(X, f)]) = \rho([(Y, g)])$. Then we have

$$[[X], ([f(\gamma_1)], \ldots, [f(\gamma_k)])] = [[Y], ([g(\gamma_1)], \ldots, [g(\gamma_k)])]$$

and thus there exists $h : X \longrightarrow Y$ an orientation preserving isometry such that $h_*([f(\gamma_i)]) = [g(\gamma_i)$ for all $i = 1, \ldots, k$. Consider the two homeomorphisms

$$\begin{aligned} h \circ f :& \Sigma_{g,n} \longrightarrow Y, \\ g :& \Sigma_{g,n} \longrightarrow Y. \end{aligned}$$

We know that there exists generally an $l \in \mathrm{Mod}_{g,n}$ such that $h \circ f$ is homotopic to $g \circ l$. Since they are homotopic they induce the same map on the free homotopy classes. Furthermore since g is a homeomorphism g_* is a bijection and thus has an inverse. Thus we get

$$g_*([\gamma_i]) = h_* f_*([\gamma_i]) = g_* l_*([\gamma_i])$$

and therefore $l_*([\gamma_i]) = [\gamma_i]$ for all $i = 1, \dots, k$ and thus $l \in G_\Gamma$, i.e. $[(X, f)] = [(Y, g)]$.

Surjectivity Pick a representative for some element in $\mathcal{M}^\Gamma_{g,n}(L)$, say (X, η) with a hyperbolic surface X and $\eta \in \mathcal{O}^X_\Gamma$. Now choose (X, f) with $f : \Sigma_{g,n} \longrightarrow X$ such that $f(\gamma_i) = \eta_i$, which is possible by construction. Then we have

$$\rho([X, f]) = [X, ([f(\gamma_1)], \dots, [f(\gamma_k)])] = [X, (\eta_1, \dots, \eta_k)].$$

The construction of the orbifold structure of $\mathcal{M}_{g,n}(L)$ is such that this map is smooth by definition. □

Now we define the projection of this set on the moduli space.

Definition 4.11. The projection $\pi^\Gamma : \mathcal{M}_{g,n}(L)^\Gamma \longrightarrow \mathcal{M}_{g,n}(L)$ is defined via

$$[X, \eta] \longmapsto [X].$$

One could hope that this space, as it is a covering of the moduli space, may be a manifold and not just an orbifold. However, this is not the case in general. Let us determine the condition when a point in $\mathcal{T}_{g,n}(L)$ has non-trivial stabilizer in G_Γ. Suppose $g \in \mathrm{Mod}_{g,n}$ fixes $(X, f) \in \mathcal{T}_{g,n}(L)$. This means that $(X, f) = (X, f \circ g^{-1})$, i.e. $f \circ g \circ f^{-1}$ is homotopic to an isometry $\phi : X \longrightarrow X$. Since $\phi_{gh} \sim f \circ g \circ h \circ f^{-1} =\sim f \circ g \circ f^{-1} \circ f \circ h \circ f^{-1} \sim \phi_g \circ \phi_h$ and $\phi \sim \mathrm{id}$ implies $g_* = \mathrm{id}$ we see that this gives a group isomorphism $\mathrm{Stab}(X, f) \simeq \mathrm{Isom}^+(X)$. So orbifold points of $\mathcal{M}_{g,n}(L)$ correspond to surfaces which have non-trivial automorphisms. However, we are only interested in the subset $G_\Gamma \subset \mathrm{Mod}_{g,n}$. If we are looking for fixed points of the G_Γ-action we have the additional condition that $f \circ g \circ f^{-1} \sim \phi$ with g fixing the curves $\gamma_1, \dots, \gamma_k$. This implies that ϕ fixes $f(\gamma_1), \dots, f(\gamma_k)$ because $\phi_*([f(\gamma_i)]) = f_* g_*([\gamma_i]) = f_*([\gamma_i])$. Thus we have non-trivial stabilizers at those points $(X, f) \in \mathcal{T}_{g,n}(L)$, for which there exists a non-trivial automorphism fixing the curves $f(\gamma_1), \dots, f(\gamma_k)$. Such points exist because one can take two surfaces with each one boundary of the same length, wich have non-trivial automorphisms fixing the special boundary and glue them together along this boundary of common length. The resulting space admits a non-trivial automorphism which fixes the glued curve (as the individual diffeomorphisms fixed the boundary) and thus may give an orbifold point on the quotient. However, for higher genus this is generically not the case and for $(g, n) = (1, 1)$ we will see in Section 4.3.1 that we obtain in fact a manifold because on a torus with one boundary there are only Dehn twists around the curve itself which fix the curve and are non-trivial in $\mathrm{Mod}_{1,1}$.

4.1.2 Symplectic reduction

When calculating the volume of the moduli space $\mathcal{M}_{g,n}(L)$ we will pass to the cover $\mathcal{M}_{g,n}(L)^\Gamma$ and then use γ and its length in order to write the integral as an integral over the lengths of γ and the corresponding reduced space. This will give us later the equations we need for proving the recurrence relation for the volumes of the moduli space. However, first we need to investigate the Weil–Petersson symplectic structure on the moduli space and look at its moment map and compute the Marsden–Weinstein reduction.
Let us begin with the construction of a function on $\mathcal{M}_{g,n}(L)^\Gamma$ which will later be the moment map of a – still to be defined – T^k-action on $\mathcal{M}_{g,n}(L)^\Gamma$.
Pick $[X, \eta] \in \mathcal{M}_{g,n}(L)^\Gamma$, where Γ is a k-tupel of good geodesics in different homotopy classes on $\Sigma_{g,n}$. Since $\mathrm{Mod}_{g,n}$ acts on each entry in $\eta \in \mathcal{O}_\Gamma$ seperately, the curves η on X are all good. Thus

we can complete them to a set of decomposing curves and use them to define Fenchel–Nielsen coordinates on $\mathcal{T}_{g,n}(L)$ for preimages of $[X]$. We will show that the associated twist flow descends to a torus action on the quotient $\mathcal{T}_{g,n}(L)\diagup G_\Gamma$.

Lemma 4.12. *The twist flow with respect to the curves in η induces a T^k-action on $\mathcal{M}_{g,n}(L)^\Gamma$.*

Proof. First, consider the twist flow tw_i^t on the Teichmüller space $\mathcal{T}_{g,n}(L)$, i.e. complete the curves $\gamma_1, \dots, \gamma_k$ to a set of decomposing curves and consider the Fenchel–Nielsen coordinates l_i, τ_i for $i = 1, \dots, 3g-3+n$. The flow of the coordinate vector fields $\frac{\partial}{\partial \tau_i}$ is given by linear functions in the τ_i-coordinate. This is the twist flow tw_i^t, see Figure 3.8. It can be imagined by cutting the surface with its hyperbolic structure along γ_i, twisting it by t and then regluing it. This is a smooth $\mathbb{R}$-action for each $i = 1, \dots, k$ on Teichmüller space. Since the G_Γ-action on $\mathcal{T}_{g,n}(L)$ stabilizes each curve γ_i around which we twist, the twist flow is equivariant with respect to this G_Γ-action. Thus it descends to a smooth action in the orbifold sense on the quotient $\mathcal{T}_{g,n}(L)\diagup G_\Gamma \simeq \mathcal{M}_{g,n}^\Gamma(L)$. However, since a twist around γ_i by $l_X(\gamma_i)$ corresponds to a Dehn twist around γ_i which is an element of G_Γ (because the γ_i are good curves and thus cannot intersect) it is identical to the identity. Thus we obtain for each curve γ_i a circle action on the quotient. Furthermore, since the curves do not intersect their twisting flows commute and one obtains a T^k-action on $\mathcal{M}_{g,n}^\Gamma(L)$. $\square$

Remark 4.13. As the lengths of the curves γ_i for $i = 1, \dots, k$ may be different we obtain a torus $\prod_{i=1}^k \mathbb{R}\diagup 2\pi l_X(\gamma_i)$. Thus we normalize the twist flow with respect to its length i.e. we consider the circle-actions $t \longmapsto \mathrm{tw}_{\gamma_i}^{t l_X(\gamma_i)}$. Now they all range from 0 to 1 and we have a standard torus.

On the Teichmüller space $\mathcal{T}_{g,n}(L)$ we have the length function $l_\Gamma : \mathcal{T}_{g,n}(L) \longrightarrow \mathbb{R}_+^k$, given by $X \longmapsto (l_{\gamma_1}(X), \dots, l_{\gamma_k}(X))$. Since it is invariant under G_Γ it descends to a function on $\mathcal{T}_{g,n}(L)\diagup G_\Gamma \simeq \mathcal{M}_{g,n}(L)^\Gamma$.

Definition 4.14. The function $\mathcal{L}_\Gamma$ is defined by

$$\begin{aligned} \mathcal{L}_\Gamma : \mathcal{M}_{g,n}^\Gamma(L) &\longrightarrow \mathbb{R}_+^k \\ [X, (\eta_1, \dots, \eta_k)] &\longmapsto (l_{\eta_1}(X), \dots, l_{\eta_k}(X)). \end{aligned}$$

Correspondingly, the i-th component of $\mathcal{L}_\Gamma$ is denoted by $\mathcal{L}_i$. It is easy to see that it is indeed the induced length function by looking at the diffeomorphism 4.2 and the following diagram.

$$\begin{tikzcd} (\mathcal{T}_{g,n}(L), \overline{\omega}_{\mathrm{WP}}) \arrow[r, "\overline{\pi}"] \arrow[rr, bend left, "\pi"] \arrow[dr, dotted, "\overline{\mathcal{L}}_\Gamma"'] & (\mathcal{M}_{g,n}^\Gamma(L), \pi^{\Gamma *}\omega_{\mathrm{WP}}) \arrow[r, "\pi^\Gamma"] \arrow[d, "\mathcal{L}_\Gamma"] & (\mathcal{M}_{g,n}(L), \omega_{\mathrm{WP}}) \\ & \mathbb{R}_+^k & \end{tikzcd} \tag{4.4}$$

Now we define a symplectic structure in the orbifold sense on $\mathcal{M}_{g,n}^\Gamma(L)$ and show that $\mathcal{L}_\Gamma$ is the moment map of the T^k-action. Explicit calculations will be done on the manifold $\mathcal{T}_{g,n}(L)$.
Since the Weil–Petersson symplectic structure $\overline{\omega}_{\mathrm{WP}}$ on $\mathcal{T}_{g,n}(L)$ is invariant under the $\mathrm{Mod}_{g,n}$ action and since $G_\Gamma \subset \mathrm{Mod}_{g,n}$ it descends to $\mathcal{T}_{g,n}(L)\diagup G_\Gamma \simeq \mathcal{M}_{g,n}^\Gamma(L)$ and is equal to the pull back of ω_{WP} on the moduli space $\mathcal{M}_{g,n}(L)$ under π^Γ, see Diagram 4.4.

Definition 4.15. The symplectic structure on $\mathcal{M}_{g,n}^\Gamma(L)$ is given by $\pi^{\Gamma *}\omega_{\mathrm{WP}}$, where ω_{WP} is the induced Weil–Petersson symplectic form on the moduli space.

By Wolperts result [39] we have that in Fenchel–Nielsen coordinates on $\mathcal{T}_{g,n}(L)$, where $\tau_i \longmapsto \tau_i + l_i$ corresponds to the Dehn twist around γ_i, $\pi^*\omega_{\mathrm{WP}} = \sum_{i=1}^{3g-3+n} \mathrm{d}l_i \wedge \mathrm{d}\tau_i$. However, as we consider the length-normalized twist-flow we need to scale τ_i such that it ranges from 0 to 1, i.e. $\tau_i = \theta_i l_i$.

Then we have $\pi^*\omega_{\mathrm{WP}} = \sum_{i=1}^{k} l_i \mathrm{d}l_i \wedge \mathrm{d}\theta_i + \sum_{i=k+1}^{3g-3+n} \mathrm{d}l_i \wedge \mathrm{d}\tau_i$, where the first k coordinates are measured with respect to the curves that we twist around, see next proof.

Lemma 4.16. *The function $\frac{1}{2}\mathcal{L}_\Gamma^2 : \mathcal{M}_{g,n}^\Gamma(L) \longrightarrow \mathbb{R}_+^k$ is the moment map for the (length-normalized) T^k action on $\mathcal{M}_{g,n}^\Gamma(L)$.*

Proof. On $\mathcal{T}_{g,n}(L)$ we can choose Fenchel–Nielsen coordinates such that they are calculated with respect to the curves γ_i. This is possible because in order to use Fenchel–Nielsen coordinates we need to specify a system of decomposing curves. However, since the γ_i's are disjoint and in different homotopy classes they can be completed to a set of decomposing curves. We consider the $\mathbb{R}^k$-action on $\mathcal{T}_{g,n}(L)$ which we used to define the torus action on the quotient $\mathcal{M}_{g,n}^\Gamma(L)$. The function $\overline{\mathcal{L}}_\Gamma : \mathcal{T}_{g,n}(L) \longrightarrow \mathbb{R}_+^k$ is given explicitely by $(X, f) \longmapsto (l_{\gamma_1}(X), \ldots, l_{\gamma_k}(X))$. The fundamental vector field for a basis vector $e_i \in \mathbb{R}^k$ is $(\mathrm{tw}_{\gamma_i}^{t l_{\gamma_i}(\cdot)})^{\cdot}(0) = l_{\gamma_i} \frac{\partial}{\partial \tau_i} = \frac{\partial}{\partial \theta_i}$. Thus we see

$$\mathrm{d}\left(\frac{1}{2}\overline{\mathcal{L}}_\Gamma^2 \cdot e_i\right) = l_{\gamma_i}\mathrm{d}l_{\gamma_i} = -\mathrm{i}_{\frac{\partial}{\partial\theta_{\gamma_i}}}\left(\sum_{j=1}^{k} l_j \mathrm{d}l_j \wedge \mathrm{d}\theta_j + \sum_{j=k+1}^{3g-3+n} \mathrm{d}l_j \wedge \mathrm{d}\tau_j\right) = -\mathrm{i}_{\frac{\partial}{\partial\tau_{\gamma_i}}}\overline{\omega}_{\mathrm{WP}}.$$

We need to check equivariance of the map $\frac{1}{2}\overline{\mathcal{L}}_\Gamma^2$. Since $\mathbb{R}^k$ is abelian the coadjoint action is trivial and we have to check that $\overline{\mathcal{L}}_\Gamma$ is invariant under the $\mathbb{R}^k$-action. However, this is immediate, because twists around the curves γ_i do not change their lengths. Thus we have shown that the $\mathbb{R}^k$-action on $\mathcal{T}_{g,n}(L)$ is Hamiltonian with respect to $\overline{\omega}_{\mathrm{WP}}$ and with momentum map $\frac{1}{2}\overline{\mathcal{L}}_\Gamma^2$.
Looking again at the Diagram 4.4 and recalling that all three objects descend to the quotient $\mathcal{M}_{g,n}^\Gamma(L)$ we see that the induced torus action is still Hamiltonian with respect to the induced symplectic form $\pi^{\Gamma *}\omega_{\mathrm{WP}}$ and the induced function $\frac{1}{2}\mathcal{L}_\Gamma^2$. □

Now that we have a Hamiltonian T^k-action on $\mathcal{M}_{g,n}^\Gamma(L)$ we can consider the symplectic quotient i.e.

$$\mathcal{M}_{g,n}^\Gamma(L)[a] := \mathcal{L}_\Gamma^{-1}(a) \big/ T^k.$$

We will consider level sets of $\mathcal{L}_\Gamma$ rather than of the actual moment map because they have a direct geometrical interpretation and there is a bijective correspondence between the level sets. Since the space $\mathcal{M}_{g,n}^\Gamma(L)$ is the quotient of Teichmüller space by a discrete group it is a symplectic orbifold and thus also $\mathcal{M}_{g,n}^\Gamma(L)[a]$ is a symplectic orbifold for all $a \in \mathbb{R}_+^k$.
Next we want to cut the surface X along the curves η_i and find a symplectomorphism between the moduli space of the cutted surface and the symplectic quotient. Therefore we define the following map

Definition 4.17. Let

$$\begin{aligned} &\mathcal{T}(\Sigma_{g,n}(\Gamma), l_\Gamma = a, l_\beta = L) \\ := \{&(X, f) \mid X \text{ a possibly disconnected hyperbolic Riemann surface with the total genus} \\ &\text{of } \Sigma_{g,n}(\Gamma) \text{ and } n + 2k \text{ boundary curves } \gamma_i, \gamma_i' \text{ and } \beta_j \text{ for } i = 1, \ldots, k \text{ and } j = 1, \ldots, n \\ &\text{of lengths } l_{\gamma_i}(X) = a_i = l_{\gamma_i'}(X) \text{ and } l_{\beta_i}(X) = L_i, \\ &f : \Sigma_{g,n}(\Gamma) \longrightarrow X \text{ orientation preserving diffeomorphism}\} \big/ \sim, \end{aligned}$$

where $\Sigma_{g,n}(\Gamma)$ is the surface obtained by cutting $\Sigma_{g,n}$ along the curves γ_i. Then we define

$$s_\Gamma : l_\Gamma^{-1}(a) \subset \mathcal{T}_{g,n}(L) \longrightarrow \mathcal{T}(\Sigma_{g,n}(\Gamma), l_\Gamma = a, l_\beta = L) \tag{4.5}$$

by cutting X along $f(\gamma)$, where f is its marking, and mapping the cutted surface to its equivalence class in $\mathcal{T}(\Sigma_{g,n}(\Gamma), l_\Gamma = a, l_\beta = L)$, see Figure 4.6.

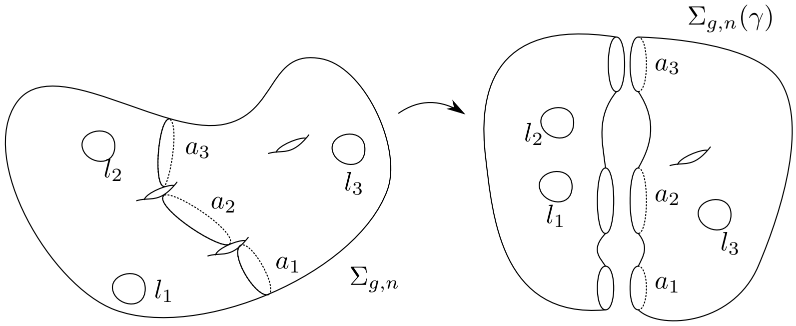


Figure 4.6: The surface $\Sigma_{g,n}$ is cutted along the curves γ_i with lengths a_i. Thus the new surface has new boundaries γ_i and γ_i' with lengths a_i which are in principle new parameters for its corresponding Teichmüller space

Since the Teichmüller space of disconnected Riemann surfaces is defined in the same way as the usual Teichmüller space the construction of the topology, the differentiable structure and the Weil–Petersson symplectic form works there, too. Since the hyperbolic structure and the marking of the individual connected components are independent of each other one has

$$\mathcal{T}(\Sigma_{g,n}(\Gamma), l_\Gamma = a, l_\beta = L) \simeq \prod_{i=1}^{s} \mathcal{T}_{g_i,n_i}(l_{A_i}),$$

where $\simeq$ means symplectomorphic, i runs over the s connected components and A_i is the set of the boundary components of the connected components, thus their length is either given by a or L. The total genus $\sum_{i=1}^{s} g_i$ is determined by the genus of the model Riemann surface $\Sigma g, n(\Gamma)$. One obtains the moduli space by taking the quotient with respect to the mapping class group of $\Sigma_{g,n}(\Gamma)$ which fixes the boundaries setwise and thus maps the individual components onto themselves. Thus we have

$$\mathcal{M}(\Sigma_{g,n}(\Gamma), l_\Gamma = a, l_\beta = L) \simeq \prod_{i=1}^{s} \mathcal{M}_{g_i,n_i}(l_{A_i}). \tag{4.6}$$

Lemma 4.18. *The map (4.5) is well defined and G_Γ-equivariant, i.e. $s_\Gamma \circ g = i_* g \circ s_\Gamma$ for $g \in G_\Gamma$ and $i : G_\Gamma \hookrightarrow Mod(\Sigma_{g,n}(\Gamma))$.*

Proof. First we need to show that if $(X, f) \sim (Y, g)$ in $\mathcal{T}_{g,n}(L)$ then $(s_\gamma(X), f_\gamma) \sim (s_\gamma(Y), g_\gamma)$ in $\mathcal{T}(\Sigma_{g,n}(\Gamma), l_\Gamma = a, l_\beta = L)$. First define the map

$$\iota : \Sigma_{g,n}(\gamma) \longrightarrow \Sigma_{g,n}$$

to be the projection map which glues the connected components togehter along γ, i.e. it is the inverse of the cutting map. Simiarly ι_X is the corresponding map for the surface X. Then, the marking of the cutted surface is obtained by setting $f_\gamma = \iota_X^{-1} \circ f \circ \iota$ outside of γ, which is a diffeomorphism outside γ and maps points close to γ again close to γ. Since $\Sigma_{g,n}(\gamma) \setminus \iota^{-1}(\gamma)$ is dense in $\Sigma_{g,n}(\gamma)$ this defines a unique continous extension to $\Sigma_{g,n}(\gamma)$ which is even smooth and an isometry. So suppose that for (X, f) and (Y, f) in $\mathcal{T}_{g,n}(L)$ there exists an isometry $\phi : X \longmapsto Y$ such that $\phi \circ f$ is isotopic to g. Now define $\overline{\phi} : s_\gamma(X) \longrightarrow s_\gamma(Y)$ by $\overline{\phi} = \iota_Y^{-1} \circ \phi \circ \iota_X$ outside of γ and extend it to γ. Then we have outside γ

$$\overline{\phi} \circ f_\gamma = \iota^{-1} \circ \phi \circ \iota_X \circ \iota_X^{-1} \circ f \circ \iota = \iota_Y^{-1} \circ \phi \circ f \circ \iota,$$

Since $\phi \circ f$ is isotopic to g we get that outside γ $\overline{\phi} \circ f_\gamma$ is isotopic to $\iota_Y^{-1} \circ g \circ \iota = g_\gamma$. Extending the isotopy to the boundary along γ we see that $(s_\gamma(X), f_\gamma) \sim (s_\gamma(Y), g_\gamma)$ in $\mathcal{T}(\Sigma_{g,n}(\Gamma), l_\Gamma = a, l_\beta = L)$. Thus the map s_γ is well defined.
Furthermore it is G_Γ equivariant due to the following argument. To show this, let $\iota_* : G_\Gamma \hookrightarrow \mathrm{Mod}_{g,n}(\Sigma_{g,n}(\gamma))$ be the induced injection of the stabilizer. It is explicitly given by $g \longmapsto \iota^{-1} \circ g \circ \iota$ outside of γ and then extended continuosly to γ. Thus we see

$$s_\gamma(g \cdot [(X, f)]) = s_\gamma([(X, f \circ g^{-1})]) = (s_\gamma(X), (f \circ g^{-1})_\gamma)$$

and since outside γ

$$(f \circ g^{-1})_\gamma = \iota_X^{-1} \circ f \circ g^{-1} \circ \iota = \iota_X^{-1} \circ f \circ \iota \circ \iota^{-1} \circ g^{-1} \circ \iota = f_\gamma \circ \iota_* g^{-1}$$

we see that $s_\gamma \circ g = \iota_* g \circ s_\gamma$ which means that s_γ is equivariant with respect to the G_Γ action. $\square$

Lemma 4.19. *The map $s_\Gamma : l_\Gamma^{-1}(a) \longrightarrow \mathcal{T}(\Sigma_{g,n}(\Gamma), l_\Gamma = a, l_\beta = L)$ induces a map*

$$s : \mathcal{M}^\Gamma_{g,n}(L)[a] \longrightarrow \mathcal{M}(\Sigma_{g,n}(\Gamma), l_\Gamma = a, l_\beta = L),$$

which is a symplectomorphism.

Proof. The map s_Γ cuts a surface along the curves γ_i, $i = 1, \dots, k$, however, at a point X in the moduli space there is no canonical representative of the curve γ on X. But since a point $[X, \eta] \subset \mathcal{M}^\Gamma_{g,n}(L)$ determines such a representative we can define a canonical map

$$s(X, \eta) = s_\Gamma(X),$$

where the right hand side is to be understood as cutting the surface X along η (which is a representative of Γ on the surface) and taking its equivalence class in the moduli space of the cutted surface. However, since

$$s_\Gamma(X) = s_\Gamma(\mathrm{tw}^t_\eta(X))$$

it is independent of the representative in the T^k-orbit on $l_\Gamma^{-1}(a)$ and thus descends to the quotient $\mathcal{M}^\Gamma_{g,n}(L)[a]$. Summarizing we have a map

$$s : \mathcal{M}^\Gamma_{g,n}(L)[a] \longrightarrow \mathcal{M}(\Sigma_{g,n}(\Gamma), l_\Gamma = a, l_\beta = L).$$

It remains to show that s is indeed a symplectomorphism. For this consider the following diagram summarizing all the introduced objects.

$$\begin{tikzcd}
(\mathcal{T}_{g,n}(L), \omega_{\mathrm{WP}}) \supset l_\Gamma^{-1}(a) \arrow[d, "\pi'"] \arrow[dr, "s_\gamma"] & & \\
(\mathcal{M}^\Gamma_{g,n}(L), \alpha) \supset \mathcal{L}_\Gamma^{-1}(a) \arrow[d, "\pi"'] & (\mathcal{T}(\Sigma_{g,n}(\gamma), l_\Gamma = a, l_\beta = L), \omega^\gamma_{\mathrm{WP}}) \arrow[r, "\simeq"] \arrow[d, "\overline{\pi}"] & \prod_{i=1}^s \mathcal{T}_{g_i,n_i}(l_{A_i}) \arrow[d] \\
(\mathcal{M}^\Gamma_{g,n}(L)[a], \overline{\alpha}) \arrow[r, "s"] & (\mathcal{M}(\Sigma_{g,n}(\gamma), l_\Gamma = a, l_\beta = L), \widehat{\alpha}) \arrow[r, "\simeq"] & \prod_{i=1}^s \mathcal{M}_{g_i,n_i}(l_{A_i})
\end{tikzcd}$$

Choosing Fenchel–Nielsen coordinates with respect to a set of decomposing curves containing $\gamma_1, \dots, \gamma_k$ we see that the map s_γ is smooth. By construction of the projections π and $\overline{\pi}$ and the orbifold structures of the moduli spaces the map s is smooth, too (in the orbifold sense). Furthermore we see that $s_\Gamma^{-1}(s_\Gamma(X)) = \{\mathrm{tw}^t_\gamma(X) \mid t \in \mathbb{R}\}$, where X denotes a point in the

Teichmüller space $\mathcal{T}_{g,n}(L)$. However, taking the quotient with respect to G_Γ and then with respect to the torus action we see that these are exactly those points which are identified when passing to the symplectic quotient $\mathcal{M}^\Gamma_{g,n}(L)[a]$. Since the lengths of the cutted curves are equal we can glue together any cutted surface in the moduli space and thus see that s is in fact bijective. Calling all the symplectic forms as in the diagram we know $\overline{\pi} \circ s_\gamma = s \circ \pi \circ \pi'$ by commutativity of the diagram, $\omega_{\mathrm{WP}} = \pi'^* \alpha$ by construction of the form on the quotient, $\alpha|_{\mathcal{L}_\Gamma^{-1}(a)} = \pi^* \overline{\alpha}$ due to the symplectic reduction and $\overline{\pi}^* \widehat{\alpha} = \omega^\gamma_{\mathrm{WP}}$ due to the construction of the symplectic form on the quotient. Here, $\pi' : l_\Gamma^{-1}(a) \longrightarrow \mathcal{L}_\Gamma^{-1}(a)$ is the projection between the level sets. Furthermore we know that $s^*_\gamma \omega^\gamma_{\mathrm{WP}} = \omega_{\mathrm{WP}}|_{l_\Gamma^{-1}(a)}$ because in both Weil–Petersson forms have in Fenchel–Nielsen coordinates with respect to a set of decomposing curves containing Γ the same form. Then we have

$$\pi'^* \pi^* s^* \widehat{\alpha} = s^*_\gamma \overline{\pi}^* \widehat{\alpha} = s^*_\gamma \omega^\gamma_{\mathrm{WP}} = \omega_{\mathrm{WP}}|_{l_\Gamma^{-1}(a)} = \pi'^* \pi^* \overline{\alpha}$$

and therefore that $s^* \widehat{\alpha} = \overline{\alpha}$ which means that s is a symplectomorphism in the orbifold sense. $\square$

4.1.3 Integration

In this section we will see how to integrate functions of the type (4.2) over the moduli space. First of all, we will integrate over open subsets of the level set $\mathcal{L}_\Gamma^{-1}(a) \subset \mathcal{M}^\Gamma_{g,n}(L)$. Call

$$\pi : \mathcal{L}_\Gamma^{-1}(a) \longrightarrow \mathcal{M}^\Gamma_{g,n}(L)[a]$$

the projection onto the symplectic quotient.

Lemma 4.20. *Let $U \subset \mathcal{M}^\Gamma_{g,n}(L)[a]$ be a sufficiently small open subset. Then*

$$Vol(\pi^{-1}(U)) = 2^{-M(\gamma)}\, Vol(U) a_1 \cdots a_k,$$

where $M(\gamma)$ denotes the number of curves γ_i which cut off a one-handle of the surface, see Figure 4.7.

Proof. Since $\pi : \mathcal{L}_\Gamma^{-1}(a) \longrightarrow \mathcal{M}^\Gamma_{g,n}(L)[a]$ is for $(g,n) \neq (1,1)$ a good T^k-orbifold principle bundle and the orbifold points are of measure zero (generic surfaces with $(g,n) \neq (1,1)$ have no non-trivial automorphisms) we can trivialize $\pi^{-1}(U) \simeq U \times T^k$ up to a set of measure zero. Thus we calculate

$$\begin{aligned}
\mathrm{Vol}(\pi^{-1}(U)) &= \int_{\pi^{-1}(U)} \pi^{\Gamma *} \mathrm{vol}_{\mathrm{WP}}|_{\pi^{-1}(U)} \\
&= \int_U \mathrm{vol}^\Gamma_{\mathrm{WP}} \int_{T^k} \mathrm{d}\tau_{\gamma_1} \ldots \mathrm{d}\tau_{\gamma_k} \\
&= \mathrm{Vol}(U) \int_0^{l_{\gamma_1}} \mathrm{d}\tau_{\gamma_1} \ldots \int_0^{l_{\gamma_k}} \mathrm{d}\tau_{\gamma_k} \\
&= \mathrm{Vol}(U) a_1 \cdots a_k 2^{-M(\gamma)}.
\end{aligned}$$

The factor of 2 appears because of the following: A generic surface with $(g,n) = (1,1)$ has an elliptic involution which interchanges the two hyperbolic hexagons that it is made of by rotating around them. This is orientation preserving and trivial in $\mathrm{Mod}_{1,1}$ as the half twist around the boundary is isotopic to the identity. However, here we have a surface attached to this boundary so we cannot just isotope it back as we would change the twisting parameter along this curve. This means that if a curve in γ, say γ_i, seperates off such a one-handle (i.e. a surface with $(g,n) = (1,1)$, see Figure 4.7) the coordinate of the twisting has only half the range, because already after $\frac{a_{\gamma_i}}{2}$

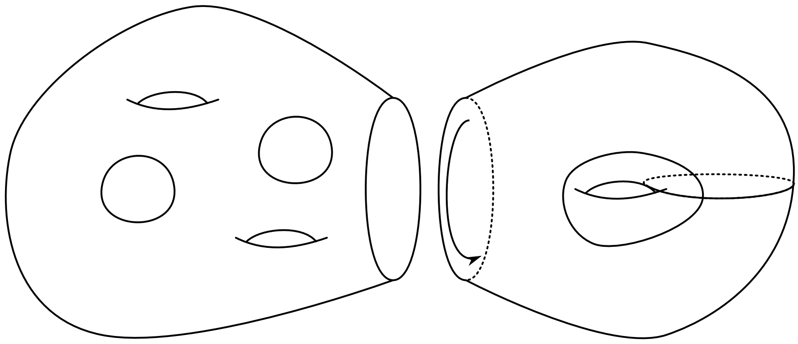


Figure 4.7: Cutting off a one-handle. By one-handle we mean a hyperbolic torus with a boundary along which it is glued to the rest of the surface. It consists of a pair of pants glued to itself, meaning that it can be built from two isometric hyperbolic hexagons. They can be obtained by cutting along the drawn lines. Thus when twisting around the gluing curve by half a rotation one interchanges the two isometric hyperbolic hexagons and thus one already arrives at the same element in the moduli space

we arrive at the same isomorphism class. Thus for each such curve the integral takes the value

$$\int_0^{\frac{a_i}{2}} \mathrm{d}\tau_{\gamma_i} = \frac{a_i}{2}$$

and we obtain correspondingly some factors of 2. □

Corollary 4.21. *We have*

$$Vol(\mathcal{L}_\Gamma^{-1}(a)) = Vol(\mathcal{M}_{g,n}^\Gamma(L)[x])a_1 \cdots a_k 2^{-M(\gamma)}.$$

Proof. Choose a partition of unity (U_i, α_i) of $\mathcal{M}_{g,n}^\Gamma(L)[a]$. Since π is surjective $(\pi^{-1}(U_i), \pi^*\alpha_i)$ is a partition of unity of $\mathcal{L}_\Gamma^{-1}(a)$. Then

$$\begin{aligned}
\mathrm{Vol}(\mathcal{L}_\Gamma^{-1}(a)) &= \sum_i \int_{\pi^{-1}(U_i)} \pi^*\alpha_i \pi^{\Gamma *}\mathrm{vol}_{\mathrm{WP}|_{\pi^{-1}(U)}} \\
&= \sum_i \int_{U_i} \alpha_i \mathrm{vol}_{\mathrm{WP}}^\Gamma \int_{T^k} \mathrm{d}\tau_{\gamma_1} \ldots \mathrm{d}\tau_{\gamma_k} \\
&= \mathrm{Vol}(\mathcal{M}_{g,n}^\Gamma(L)[a])a_1 \cdots a_k 2^{-M(\gamma)}.
\end{aligned}$$

□

Now we will state the first version of the integral equation.

Lemma 4.22. *Let $F : \mathbb{R}^k \longrightarrow \mathbb{R}_+$ be a continuous function. For $\Gamma = (\gamma_1, \ldots, \gamma_k)$ a k-tupel of curves as before define $F_\Gamma : \mathcal{M}_{g,n}^\Gamma(L) \longrightarrow \mathbb{R}$ by*

$$F_\Gamma = F \circ \mathcal{L}_\Gamma.$$

Then

$$\int_{\mathcal{M}_{g,n}^\Gamma(L)} F_\Gamma \mathrm{vol}_{\pi^{\Gamma *}\omega_{\mathrm{WP}}} = 2^{-M(\gamma)} \int_{x \in \mathbb{R}_+^k} F(x)\mathrm{Vol}(\mathcal{M}(\Sigma_{g,n}(\gamma), l_\Gamma = x, l_\beta = L))x \, \mathrm{d}x,$$

where $x \, \mathrm{d}x = x_1 \cdots x_k \, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_k$.

Proof. As we have already pointed out $\mathcal{L}_\Gamma : \mathcal{T}_{g,n}(L) \longrightarrow \mathbb{R}_+^k$ is invariant under the G_Γ-action such that it descends to a smooth (in the orbifold sense) map on the quotient. Thus we can integrate first over the level sets of $\mathcal{L}_\Gamma$ and then over its values, i.e.

$$\begin{aligned}
\int_{\mathcal{M}_{g,n}^\Gamma(L)} F_\Gamma \operatorname{vol}_{\pi^{\Gamma *}\omega_{\mathrm{WP}}} &= \int_{x\in\mathbb{R}_+^k} \left(\int_{\mathcal{L}_\Gamma^{-1}(x)} F \circ \mathcal{L}_\Gamma \operatorname{vol}_{\pi^{\Gamma *}\omega_{\mathrm{WP}}|_{\mathcal{L}_\Gamma^{-1}(x)}} \right) \mathrm{d}x \\
&= \int_{x\in\mathbb{R}_+^k} F(x)\mathrm{Vol}(\mathcal{L}_\Gamma^{-1}(x))\mathrm{d}x \\
&= 2^{-M(\gamma)} \int_{x\in\mathbb{R}_+^k} F(x)\mathrm{Vol}(\mathcal{M}_{g,n}^\Gamma(L)[x])x\mathrm{d}x \\
&= 2^{-M(\gamma)} \int_{x\in\mathbb{R}_+^k} F(x)\mathrm{Vol}(\mathcal{M}(\Sigma_{g,n}(\gamma), l_\Gamma = x, l_\beta = L))x\mathrm{d}x,
\end{aligned}$$

where we have used Lemmas 4.19 and 4.21. □

Theorem 4.23. *Let $\gamma = \sum_{i=1}^k c_i\gamma_i$ be a multicurve on $\Sigma_{g,n}$. Further define $|x| := \sum_{I=1}^k c_i x_i$ and $f_\gamma : \mathcal{M}_{g,n}(L) \longrightarrow \mathbb{R}_+$ as in Definition 4.2 for a function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$. Then*

$$\int_{\mathcal{M}_{g,n}(L)} f_\gamma \operatorname{vol}_{\omega_{\mathrm{WP}}} = \frac{2^{-M(\gamma)}}{|\mathrm{Sym}(\gamma)|} \int_{x\in\mathbb{R}_+^k} f(|x|) V_{g,n}(\Gamma, x, \beta, L) x \, \mathrm{d}x, \tag{4.7}$$

where $V_{g,n}(\Gamma, x, \beta, L) := \mathrm{Vol}(\mathcal{M}(\Sigma_{g,n}(\gamma), l_\Gamma = x, l_\beta = L))$.

Proof. Define $F : \mathbb{R}_+^k \longrightarrow \mathbb{R}_+$ by

$$(x_1, \ldots, x_k) \longmapsto f(\sum_{i=1}^k c_i x_i)$$

and $F_\Gamma : \mathcal{M}_{g,n}^\Gamma(L) \longrightarrow \mathbb{R}$ as before. Then we get in the notation of Lemma 4.7

$$\pi_*^\Gamma F_\Gamma : \mathcal{M}_{g,n}(L) \longrightarrow \mathbb{R}_+ \tag{4.8}$$

$$X \longmapsto \sum_{Y\in(\pi^\Gamma)^{-1}(X)} F_\Gamma(Y) = \sum_{Y\in(\pi^\Gamma)^{-1}(X)} f(l_\gamma(Y)). \tag{4.9}$$

Let us investigate the combinatorics of this expression a little bit further. First we have $(\pi^\Gamma)^{-1}(X) = \{X\} \times [\phi(\mathrm{Mod}_{g,n} \cdot [\gamma])]$ where ϕ is the marking and $[\ldots]$ denotes the free homotopy class on the surface. However, elements in $\mathrm{Mod}_{g,n}$ which differ by something in $\bigcap_{i=1}^k \mathrm{Stab}(\gamma_i)$ define the same element in $\mathcal{O}_\Gamma^X$. Thus we can rewrite the expression (4.8) as

$$\pi_*^\Gamma F_\Gamma = \sum_{g\in\mathrm{Mod}_{g,n}/\bigcap_{i=1}^k \mathrm{Stab}(\gamma_i)} f(l_{g\cdot\gamma}(X)).$$

Now we could instead sum over all $[\alpha]$ such that $\alpha = g\cdot\gamma$, i.e. $[\alpha] \in \mathrm{Mod}_{g,n} \cdot [\gamma]$, however, in this sum the whole stabilizer of the multicurve $\mathrm{Stab}(\gamma)$ is not counted, thus we have to compensate for that. Since $l_\alpha(X) = l_{g\cdot\alpha}(X)$ for $g \in \mathrm{Stab}(\alpha)$ we see that we just have to multiply this expression by the number of elements in $\mathrm{Stab}(\gamma)/\bigcap_{i=1}^k \mathrm{Stab}(\gamma_i)$. Summarizing we get

$$\mathrm{Mod}_{g,n} \cdot [\gamma] \simeq \frac{\mathrm{Mod}_{g,n}}{\mathrm{Stab}(\gamma)} \simeq \frac{\mathrm{Mod}_{g,n}/\bigcap_i \mathrm{Stab}(\gamma_i)}{\mathrm{Stab}(\gamma)/\bigcap_i \mathrm{Stab}(\gamma_i)} \simeq \frac{\mathrm{Mod}_{g,n}/\bigcap_i \mathrm{Stab}(\gamma_i)}{\mathrm{Sym}(\gamma)},$$

where $\simeq$ means bijective correspondence. So we obtain

$$\pi_*^\Gamma F_\Gamma = |\mathrm{Sym}(\gamma)| \sum_{[\alpha]\in \mathrm{Mod}_{g,n}\cdot[\gamma]} f\circ l_\alpha = |\mathrm{Sym}(\gamma)|\, f_\gamma.$$

Now we use Lemma 4.7 and 4.22 to see

$$\begin{aligned}\int_{\mathcal{M}_{g,n}(L)} \pi_*^\Gamma F_\Gamma \mathrm{vol}_{\omega_{WP}} &= \int_{\mathcal{M}^\Gamma_{g,n}(L)} F_\Gamma \mathrm{vol}_{\pi^{\Gamma *}\omega_{WP}} \\ &= 2^{-M(\gamma)} \int_{x\in\mathbb{R}^k_+} f(|x|) V_{g,n}(\Gamma,x,\beta,L) x \,\mathrm{d}x\end{aligned}$$

and thus

$$\int_{\mathcal{M}_{g,n}(L)} f_\gamma \mathrm{vol}_{\omega_{WP}} = \frac{2^{-M(\gamma)}}{|\mathrm{Sym}(\gamma)|}\int_{x\in\mathbb{R}^k_+} f(|x|) V_{g,n}(\Gamma,x,\beta,L) x \,\mathrm{d}x.$$

□

4.2 Recursion relation for the volumes of the moduli spaces

In this section we will use the result of the last section in order to deduce a recursion relation for the Weil–Petersson volume of moduli spaces of Riemann surfaces. The basic idea is to integrate the McShane identity (2.44) over the moduli space. Let us begin by recalling this identity.

Theorem 4.24. *For all $X \in \mathcal{T}_{g,n}(L)$ we have*

$$\sum_{\{\gamma_1,\gamma_2\}\in\mathcal{F}_1} \mathcal{D}(L_1, l_{\gamma_1}(X), l_{\gamma_2}(X)) + \sum_{i=2}^{n}\sum_{\gamma\in\mathcal{F}_{1,i}} \mathcal{R}(L_1, L_i, l_\gamma(X)) = L_1, \tag{4.10}$$

where the $\mathcal{F}_i$, $\mathcal{F}_{i,j}$, $\mathcal{D}$ and $\mathcal{R}$ are defined as follows:

$$\begin{aligned}\mathcal{F}_i &:= \{\{\gamma_1,\gamma_2\} \mid \gamma_i \textit{ are unordered free homotopy classes of non-peripheral simple} \\ &\qquad \textit{closed curves on } \Sigma_{g,n} \textit{ which bound an embedded pair of pants in } \Sigma_{g,n} \textit{ with } \beta_i\} \\ \mathcal{F}_{i,j} &:= \{\gamma \mid \gamma \textit{ is a free homotopy class of closed non-peripheral simple curves in } \Sigma_{g,n} \\ &\qquad \textit{such that } \gamma,\ \beta_i \textit{ and } \beta_j \textit{ bound an embedded pair of pants in } \Sigma_{g,n}\} \\ \mathcal{D}(x,y,z) &:= 2\ln\left(\frac{e^{\frac{x}{2}} + e^{\frac{y+z}{2}}}{e^{-\frac{x}{2}} + e^{\frac{y+z}{2}}}\right) \\ \mathcal{R}(x,y,z) &:= x - \ln\left(\frac{\cosh\left(\frac{y}{2}\right) + \cosh\left(\frac{x+z}{2}\right)}{\cosh\left(\frac{y}{2}\right) + \cosh\left(\frac{x-z}{2}\right)}\right)\end{aligned}$$

First, we will reformulate this identity slightly, afterwards investigate the sets $\mathcal{F}_i$ and $\mathcal{F}_{i,j}$ and then integrate the two terms seperatly.

Definition 4.25. Define the functions $\tilde{\mathcal{R}}_j$ and $\tilde{\mathcal{D}}$ on the moduli space $\mathcal{M}_{g,n}(L)$ by

$$\begin{aligned}\tilde{\mathcal{R}}_j(X) &= \sum_{\gamma\in\mathcal{F}_{1,j}} \mathcal{R}(L_1, L_j, l_\gamma(X)), \\ \tilde{\mathcal{D}}(X) &= \sum_{\{\gamma_1,\gamma_2\}\in\mathcal{F}_1} \mathcal{D}(L_1, l_{\gamma_1}(X), l_{\gamma_2}(X)).\end{aligned}$$

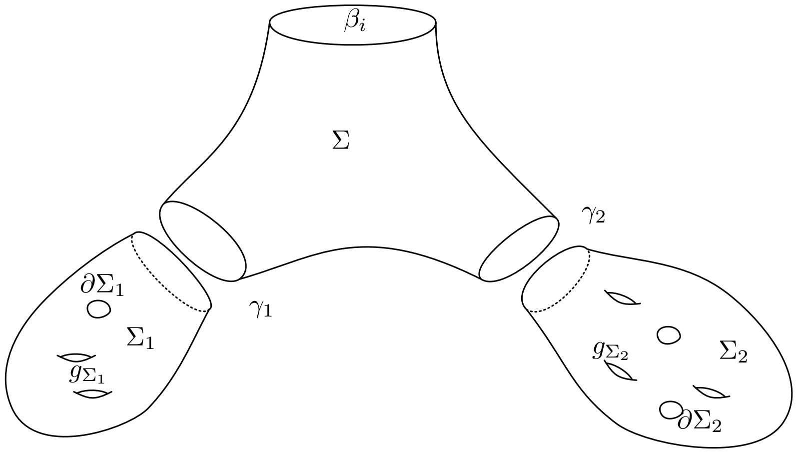


Figure 4.8: The two parts of a cut surface without the pyir of pants containing β_i

In order to see that these are well defined functions on the moduli space we need to know how $\mathrm{Mod}_{g,n}$ acts on the sets $\mathcal{F}_1$ and $\mathcal{F}_{1,j}$.

Lemma 4.26. *$\mathrm{Mod}_{g,n}$ acts on $\mathcal{F}_1$ and $\mathcal{F}_{1,j}$. It acts even transitively on $\mathcal{F}_{1,j}$ for $j \neq 1$. The set $\mathcal{F}_1$ is bijective to $A^{con} \cup A^{dcon}$, where $A^{dcon} = \bigcup_{a \in \mathcal{I}_{g,n}} A_a$ and*

$$\begin{aligned}
A^{con} &:= \{\gamma_1 + \gamma_2 \mid \{\gamma_1, \gamma_2\} \in \mathcal{F}_1 \textit{ such that the complement of their pair of pants is} \\
&\qquad \textit{a connected surface of genus } g-1 \textit{ with } n+1 \textit{ boundary components}\} \\
\mathcal{I}_{g,n} &= \{((g_1, I_1), (g_2, I_2)) \mid \textit{set of ordered pairs with } I_1, I_2 \subset \{2, \ldots, n\} \textit{ and } 0 \leq g_1, g_2 \leq g \\
&\qquad \textit{such that } I_1 \textit{ and } I_2 \textit{ are disjoint and } I_1 \cup I_2 = \{2, \ldots, n\} \\
&\qquad g_1, g_2 \textit{ and } n_i = |I_i| \textit{ satisfy } 2 \leq 2g_i + n_i \textit{ for } i = 1, 2 \textit{ and } g_1 + g_2 = g \\
A_a &:= \{\gamma_1 + \gamma_2 \mid \{\gamma_1, \gamma_2\} \subset \mathcal{F}_1 \textit{such that the complement of their bounded pair of pants is} \\
&\qquad \textit{the disjoint union of two surfaces } \Sigma_1 \textit{ and } \Sigma_2, \textit{ homeomorphic to } \Sigma_{g_1,n_1+1} \textit{ and} \\
&\qquad \Sigma_{g_2,n_2+1}, \textit{ respectively, such that } \beta_{i_1} \subset \partial\Sigma_1 \textit{ for } i_1 \in I_1 \textit{ and } \beta_{i_2} \subset \partial\Sigma_2 \textit{ for } i_2 \in I_2.
\end{aligned}$$

Furthermore Mod_n acts transitively on A^{con} and A_a for $a \in \mathcal{I}_{g,n}$.

These definitions are illustrated in Figure 4.8. Although they look rather complicated there is a simple reason we need them. By (4.7) we can express the integral in terms of volumes of moduli spaces of cutted surfaces. However, we will have to do this for all possible cuts and thus need a combinatorial way of describing them.

Proof. Proving these statements requires extensive use of Lemma 3.7, so let us recall, that there exists a diffeomorphism interchanging two good curves on a Riemann surface if and only if the surfaces obtained by cutting along thes curves are homeomorphic by a self-homeomorphism, i.e. the map must fix the boundaries setwise. The last condition can be checked e.g. by counting boundaries and holes and looking at the map at the boundary.

$\mathrm{Mod}_{g,n}$ acts on $\mathcal{F}_1$
Let $\{\gamma_1, \gamma_2\} \in \mathcal{F}_1$, see Figure 4.8, and let $g \in \mathrm{Mod}_{g,n}$. Call the pair of pants bounded by γ_1, γ_2 and β_1 Σ. $g_{|\Sigma}$ is still a homeomorphism and thus $g(\Sigma)$ is still a pair of pants bounded by $g(\partial\Sigma)$. Thus, $g(\{\gamma_1, \gamma_2\}) \in \mathcal{F}_1$.

$\mathrm{Mod}_{g,n}$-action on $\mathcal{F}_{1,j}$ and its transitivity
Let $\gamma \in \mathcal{F}_{1,j}$ be a good curve bounding the pair of pants Σ together with β_1 and β_j, see Figure 4.9b. Again, $g_{|\Sigma}$ is a homeomorphism and fixes β_1 and β_j and maps Σ to another pair of pants. Thus

$g(\gamma)$ bounds again a pair of pants with β_1 and β_j. Furthermore we see that for any $\gamma \in \mathcal{F}_{1,j}$ the cut surface $\Sigma_{g,n}(\gamma)$ is homeomorphic to the disjoint union of a pair of pants and the remaining surface, i.e. $\Sigma_{g,n}(\gamma) \simeq \Sigma_{0,3} \cup \Sigma_{g,n-1}$. Since the boundaries of all the pairs are the same (except for γ) all these cut surfaces are homeomorphic by a self-homeomorphism, and thus by Lemma 3.7 $\mathrm{Mod}_{g,n}$ acts transitively on $\mathcal{F}_{1,j}$.

Bijectivity of $\mathcal{F}_1$ and $\mathcal{A}^{con} \cup \mathcal{A}^{dcon}$
$\mathcal{A}^{\mathrm{con}}, \mathcal{A}^{\mathrm{dcon}} \in \mathcal{F}_1$ is obvious by definition. Take any $\{\gamma_1, \gamma_2\} \in \mathcal{F}_1$ with bounded pair of pants Σ. For the cut surface $\Sigma_{g,n}(\gamma)$ there are exactly two possibilities. Either it is homeomorphic to $\Sigma \cup \Sigma_{g-1,n+1}$ if the remaining surface is connected or it is disconnected and thus homeomorphic to $\Sigma \cup \Sigma_{g_1,n_1+1} \cup \Sigma_{g_2,n_2+1}$. In the first case we have $\{\gamma_1, \gamma_2\} \in \mathcal{A}^{\mathrm{con}}$. In the second case we see by looking at Figure 4.8 that $g_1 + g_2 = g$, $n_1 + n_2 = n - 1$ and

$$\begin{aligned} \partial\Sigma_{g_1,n_1+1} &= \beta_{i_1} \cup \ldots \cup \beta_{i_{n_1}} \cup \gamma_1, \\ \partial\Sigma_{g_2,n_2+1} &= \beta_{j_1} \cup \ldots \cup \beta_{j_{n_2}} \cup \gamma_2, \end{aligned}$$

where for the index sets one has $I_1 \sqcup I_2 = \{2, \ldots, n\}$. Furthermore since γ_1 and γ_2 are non-peripheral and essential the connected cut surfaces Σ_{g_i,n_i+1} are stable, i.e. $2g_i - 2 + n_i + 1 > 0$. This implies $2 \leq 2g_i + n_i$. Thus, $\{\gamma_1, \gamma_2\} \in \mathcal{A}_a$ for some $a \in \mathcal{I}_{g,n}$. The two sets are thus equal.

$Mod_{g,n}$-action on $\mathcal{A}^{con}$
Let $\{\gamma_1, \gamma_2\}$ be such that the cut surface is homeomorphic to a pair of pants Σ union a connected Riemann surface $\Sigma_{g-1,n+1}$. Given any $g \in \mathrm{Mod}_{g,n}$ we see as above that the image of the cut surface components under g is again such that $g(\gamma_1), g(\gamma_2)$ and β_1 bound an embedded pair of pants and that $g(\Sigma_{g-1,n+1})$ is still connected. Thus $\mathrm{Mod}_{g,n}$ acts on $\mathcal{A}^{\mathrm{con}}$. Furthermore, since for all such curves γ_1 and γ_2 the cut surfaces are homeomorphic and contain the same boundaries except for γ_1 and γ_2, $\mathrm{Mod}_{g,n}$ acts transitively on $\mathcal{A}^{\mathrm{con}}$.

$Mod_{g,n}$-action on $\mathcal{A}_a$
For a fixed $a \in \mathcal{I}_{g,n}$ let $\{\gamma_1, \gamma_2\} \in \mathcal{A}_a$ be arbitrary. Then we have that the cut surface $\Sigma_{g,n}(\gamma) \simeq \Sigma_{0,3} \cup \Sigma_{g_1,n_1+1} \cup \Sigma_{g_2,n_2+1}$ holds for all such γ. a even fixes which boundaries are on which of the connected components of the cut surface. Thus we have again by the same arguing as before that $\mathrm{Mod}_{g,n}$ acts transitively on $\mathcal{A}_a$ for each $a \in \mathcal{I}_{g,n}$. □

Corollary 4.27. *The functions $\tilde{\mathcal{R}}_j$ and $\tilde{\mathcal{D}}$ are well defined functions on $\mathcal{M}_{g,n}(L)$ and satisfy*

$$\tilde{\mathcal{D}}(X) + \sum_{j=2}^{n} \tilde{\mathcal{R}}_j(X) = L_1. \tag{4.11}$$

Proof. Since the sums in the definitions of $\tilde{\mathcal{R}}_j$ and $\tilde{\mathcal{D}}$ range over all curves of this kind and since Mod_n acts on the sets $\mathcal{F}_i$ and $\mathcal{F}_{i,j}$ a change of a representative of $X \in \mathcal{M}_{g,n}(L)$ only changes the order in the summation. Thus the functions are well defined. Furthermore (4.11) is just a rewriting of (4.10). □

Definition 4.28. For convenience we define the following objects

$$\begin{aligned} H: \quad & \mathbb{R}^2 \longrightarrow \mathbb{R},\ (x,y) \longmapsto \frac{1}{1+e^{\frac{x+y}{2}}} + \frac{1}{1+e^{\frac{x-y}{2}}}, \\ m(g,n) &= \delta_{g,1}\delta_{n,1}, \\ V_{g,n}(L) &= \int_{\mathcal{M}_{g,n}(L)} \mathrm{vol}_{\mathrm{WP}}. \end{aligned}$$

Remark 4.29. Here, $\delta_{i,j}$ is the Kronecker delta symbol, i.e. it is one if $i = j$ and zero otherwise. This term will appear frequently due to the existence of non-trivial automorphisms of once-bordered tori, i.e. for $g = n = 1$. The function H will appear quite often as an integral kernel due to the relations in the next lemma.

Lemma 4.30. *We have*

$$\frac{\partial}{\partial x}\mathcal{D}(x,y,z) = H(y+z,x), \tag{4.12}$$
$$\frac{\partial}{\partial x}\mathcal{R}(x,y,z) = \frac{1}{2}(H(z,x+y)+H(z,x-y)). \tag{4.13}$$

Proof. The first derivative is straight-forward.

$$\begin{aligned}\frac{\partial}{\partial x}\mathcal{D}(x,y,z) &= 2\frac{e^{\frac{-x}{2}}+e^{\frac{y+z}{2}}}{e^{\frac{x}{2}}+e^{\frac{y+z}{2}}}\left(\frac{e^{\frac{x}{2}}}{2(e^{\frac{-x}{2}}+e^{\frac{y+z}{2}})}+\frac{e^{\frac{-x}{2}}(e^{\frac{x}{2}}+e^{\frac{y+z}{2}})}{2(e^{\frac{-x}{2}}+e^{\frac{y+z}{2}})^2}\right)\\ &= \frac{1}{1+e^{\frac{y+z-x}{2}}}+\frac{1}{1+e^{\frac{y+z+x}{2}}}\\ &= H(y+z,x).\end{aligned}$$

Using Theorem 2.31 it is also easy to derive the other formula

$$\begin{aligned}\frac{\partial}{\partial x}\mathcal{R}(x,y,z) &= \frac{1}{2}\frac{\partial}{\partial x}\left(\mathcal{D}(x,y,z)+\mathcal{D}(x,-y,z)\right)\\ &= \frac{1}{2}(H(y+z,x)+H(z-y,x))\\ &= \frac{1}{2}(H(z,x+y)+H(z,x-y)),\end{aligned}$$

where the last identity is easily checked. □

Lemma 4.31. *For the second term in (4.11) we have*

$$\begin{aligned}&\sum_{j=2}^{n}\frac{\partial}{\partial L_1}\int_{\mathcal{M}_{g,n}(L)}\tilde{\mathcal{R}}_j \operatorname{vol}_{\mathrm{WP}}\\ =&\frac{2^{-m(g,n-1)}}{2}\sum_{j=2}^{n}\int_0^\infty x\left(H(x,L_1+L_j)+H(x,L_1-L_j)\right)V_{g,n-1}(x,L_2,\ldots,\widehat{L_j},\ldots,L_n)\mathrm{d}x\end{aligned} \tag{4.14}$$

Proof. Fix a $j\in\{2,\ldots,n\}$. Then by Lemma 4.26 $\mathrm{Mod}_{g,n}$ acts transitively on $\mathcal{F}_{1,j}$. Thus we can fix a free homotopy class γ_j on $\Sigma_{g,n}$ such that

$$\mathrm{Mod}_{g,n}\cdot\gamma_j = \mathcal{F}_{1,j}. \tag{4.15}$$

Furthermore define a function $R^j:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ by

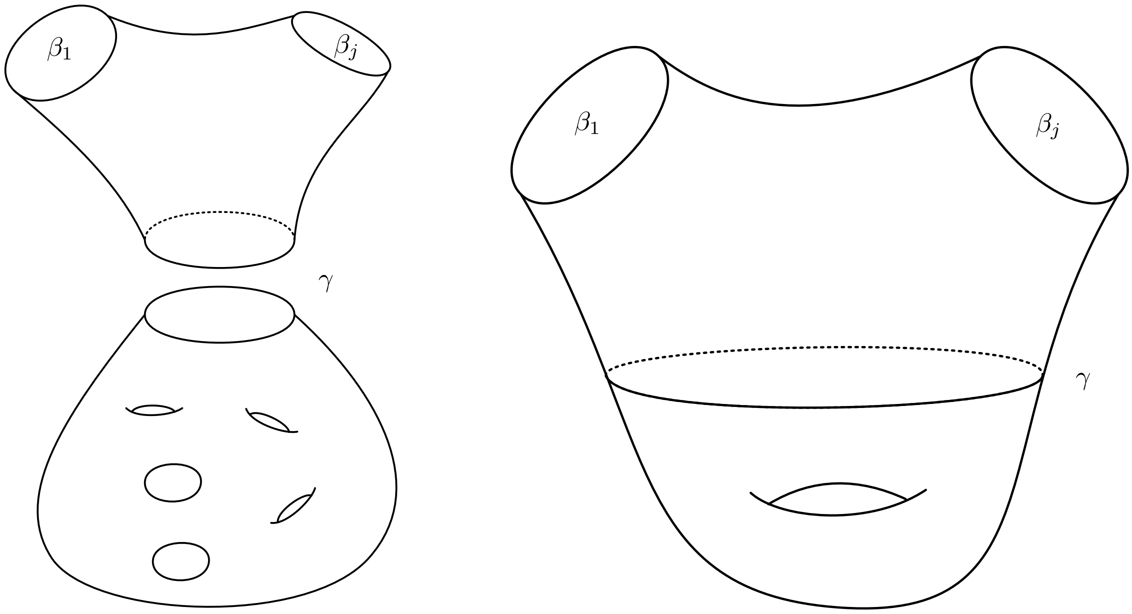
$$R^j(x) := \mathcal{R}(L_1,L_j,x).$$

Now we can see that

$$\tilde{\mathcal{R}}_j(X) = \sum_{\gamma\in\mathcal{F}_{1,j}}\mathcal{R}(L_1,L_j,l_\gamma(X)) = R^j_{\gamma_j}(X)$$

by (4.15) and Definition 4.2. Now we apply (4.7) in order to obtain

$$\int_{\mathcal{M}_{g,n}(L)}\tilde{\mathcal{R}}_j\operatorname{vol}_{\mathrm{WP}} = \int_{\mathcal{M}_{g,n}(L)}R^j_{\gamma_j}\operatorname{vol}_{\mathrm{WP}} = \frac{2^{-M(\gamma_j)}}{|\mathrm{Sym}(\gamma_j)|}\int_0^\infty\mathcal{R}(L_1,L_j,x)V_{g,n}(\gamma_j,x,\beta,L)x\,\mathrm{d}x.$$



(a) $\Sigma_{g,n}(\gamma_j) \simeq \Sigma_{g,n-1}\dot{\cup}\Sigma_{1,1}$ (b) The case $(g,n) = (1,2)$

Figure 4.9

By looking at Figure 4.9a we see $\Sigma_{g,n}(\gamma_j) \simeq \Sigma_{g,n-1} \cup \Sigma_{1,1}$. Because $V_{g,n}(\gamma_j, x, \beta, L)$ denotes the volume of the moduli space of the cutted surface and because the cutted surface is actually homeomorphic to $\Sigma_{g,n-1}$ union a hyperbolic pair of pants which gives vol(pt) = 1 we see that the volume is actually given by $V_{g,n-1}(x, L_2, \ldots, \hat{L_j}, \ldots, L_n)$. Furthermore, $|\mathrm{Sym}(\gamma_j)| = 1$ and since γ_j always seperates the surface we have $M(\gamma_j) = m(g, n-1)$, see Figure 4.9b. Thus we can rewrite the last equation as

$$\int_{\mathcal{M}_{g,n}(L)} \tilde{\mathcal{R}}_j \,\mathrm{vol}_{\mathrm{WP}} = 2^{-m(g,n-1)} \int_0^\infty x\mathcal{R}(L_1, L_j, x) V_{g,n-1}(x, L_2, \ldots, \hat{L_j}, \ldots, L_n)\mathrm{d}x.$$

Now we apply (4.13) and get the final result

$$\begin{aligned}&\sum_{j=2}^{n} \frac{\partial}{\partial L_1} \int_{\mathcal{M}_{g,n}(L)} \tilde{\mathcal{R}}_j \,\mathrm{vol}_{\mathrm{WP}} \\ =&\frac{2^{-m(g,n-1)}}{2} \sum_{j=2}^{n} \int_0^\infty x \left(H(x, L_1 + L_j) + H(x, L_1 - L_j)\right) V_{g,n-1}(x, L_2, \ldots, \hat{L_j}, \ldots, L_n)\mathrm{d}x\end{aligned}$$

□

Lemma 4.32. *For the first term in (4.11) we have*

$$\begin{aligned}&\frac{\partial}{\partial L_1} \int_{\mathcal{M}_{g,n}(L)} \tilde{\mathcal{D}} \,\mathrm{vol}_{\mathrm{WP}} \\ &\quad= \frac{1}{2} \int_0^\infty \int_0^\infty xy \left(\frac{V_{g-1,n+1}(x, y, L_2, \ldots, L_n) H(x+y, L_1)}{2^{m(g-1,n+1)}} + \right. \\ &\qquad\left. \sum_{a \in \mathcal{I}_{g,n}} \frac{V_{g_1,n_1+1}(x, L_{I_1})}{2^{m(g_1,n_1+1)}} \frac{V_{g_2,n_2+1}(y, L_{I_2})}{2^{m(g_2,n_2+1)}} H(x+y, L_1) \right) \mathrm{d}x \,\mathrm{d}y\end{aligned} \tag{4.16}$$

Proof. From the definition of $\mathcal{D}$ in Theorem 2.44 we can see that $\mathcal{D}(x,y,z)=\mathcal{D}(x,y+z,0)$. Now define $D:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ by

$$D(x):=\mathcal{D}(L_1,x,0).$$

By Lemma 4.26 we know what the set $\mathcal{F}_1$ looks like. Choose a representative for each equivalence class in $\mathcal{F}_1/\mathrm{Mod}_{g,n}$, i.e. $\mathcal{C}=\{\gamma_a^1+\gamma_a^2 \mid a\in\mathcal{I}_{g,n}\}\cup\{\gamma\}$ is bijective to the set of equivalence classes. Then we have

$$\begin{aligned}\tilde{\mathcal{D}}(X)&=\sum_{\{\gamma_1,\gamma_2\}\in\mathcal{F}_1}\mathcal{D}(L_1,l_{\gamma_1}(X),l_{\gamma_2}(X))=\sum_{\gamma=\gamma_1+\gamma_2\in\mathcal{C}}\ \sum_{[\alpha]\in\mathrm{Mod}_{g,n}\cdot[\gamma]}\mathcal{D}(L_1,l_\alpha(X),0)\\&=\sum_{\gamma=\gamma_1+\gamma_2\in\mathcal{C}}D_\gamma(X).\end{aligned}$$

Now we can use the integration formula (4.7) to obtain

$$\begin{aligned}\frac{\partial}{\partial L_1}\int_{\mathcal{M}_{g,n}(L)}\tilde{\mathcal{D}}\ \mathrm{vol_{WP}}&=\frac{\partial}{\partial L_1}\sum_{\gamma=\gamma_1+\gamma_2\in\mathcal{C}}\int_{\mathcal{M}_{g,n}(L)}D_\gamma\ \mathrm{vol_{WP}}\\&=\frac{\partial}{\partial L_1}\sum_{\gamma=\gamma_1+\gamma_2\in\mathcal{C}}\frac{2^{-M(\gamma)}}{|\mathrm{Sym}(\gamma)|}\int_{x\in\mathbb{R}_+^2}\mathcal{D}(L_1,x_1+x_2,0)V_{g,n}(\Gamma,x,\beta,L)x\ \mathrm{d}x\\&=\sum_{\gamma=\gamma_1+\gamma_2\in\mathcal{C}}\frac{2^{-M(\gamma)}}{|\mathrm{Sym}(\gamma)|}\int_0^\infty\int_0^\infty xyH(x+y,L_1)V_{g,n}(\Gamma,(x,y),\beta,L)\mathrm{d}x\mathrm{d}y,\end{aligned}\tag{4.17}$$

where we have used (4.12). Now it remains to split this sum in the two subsets of $\mathcal{C}$ and use Lemma 4.6 and Figure 4.9b to express this equation in the wanted form.
The sum over $\mathcal{C}$ can be written as the sum of the term for the representative of A^{con} and the terms for the A_a, where $a\in\mathcal{I}_{g,n}$.
1. Case – A^{con} By Lemma 4.6 we see that $|\mathrm{Sym}(\gamma)|=2$. The cut off surface has genus $g-1$ and $n+1$ boundary components. Thus, $V_{g,n}(\Gamma,(x,y),\beta,L)=V_{g-1,n+1}(x,y,L_2,\dots,L_n)$. Furthermore, γ_1 and γ_2 seperate off a one-handle if and only if $g-1=1$ and $n+1=1$.
2. Case – A_a Let $a=((g_1,I_1),(g_2,I_2))$. Then γ seperates off a 1-handle if $g_1=1, n_1+1=1$ or $g_2=1, n_2+1=1$. $|\mathrm{Sym}(\gamma)|=2$ if and only if $g_1=g_2$ by Lemma 4.6. However, if we write the sum over $\mathcal{C}$ in terms of $a\in\mathcal{I}_{g,n}$ which were defined as ordered pairs, we count every summand twice, except those for which there is no reversed pair. These are the elements for which $I_1=I_2=\emptyset$ and $g_1=g_2$, i.e. exactly those terms for which we get the factor $\frac{1}{2}$ due to $|\mathrm{Sym}(\gamma)|$. All in all we see that we obtain an overall factor of $\frac{1}{2}$. Furthermore, by cutting the surface we obtain $V_{g,n}(\Gamma,(x,y),\beta,L)=V_{g_1,n_1+1}(x,L_{I_1})V_{g_2,n_2+1}(y,L_{I_2})$.
To summarize, we can write (4.17) as

$$\begin{aligned}&\frac{\partial}{\partial L_1}\int_{\mathcal{M}_{g,n}(L)}\tilde{\mathcal{D}}\ \mathrm{vol_{WP}}\\&\quad=\frac{1}{2}\int_0^\infty\int_0^\infty xy\left(\frac{V_{g-1,n+1}(x,y,L_2,\dots,L_n)H(x+y,L_1)}{2^{m(g-1,n+1)}}+\right.\\&\qquad\left.\sum_{a\in\mathcal{I}_{g,n}}\frac{V_{g_1,n_1+1}(x,L_{I_1})}{2^{m(g_1,n_1+1)}}\frac{V_{g_2,n_2+1}(y,L_{I_2})}{2^{m(g_2,n_2+1)}}H(x+y,L_1)\right)\mathrm{d}x\ \mathrm{d}y\end{aligned}$$

□

Now we put everything together to get the recursion relation of the Weil–Petersson volumes.

Theorem 4.33. *For the Weil–Petersson volume $V_{g,n}(L)$ of the moduli space of bordered Riemann surfaces $\mathcal{M}_{g,n}(L)$ for $(g,n) \neq (1,1),(0,3)$ we have*

$$\begin{aligned}
&\frac{\partial}{\partial L_1} L_1 V_{g,n}(L) = \\
&\quad \frac{1}{2}\int_0^\infty \int_0^\infty xy \left(\frac{V_{g-1,n+1}(x,y,L_2,\dots,L_n)H(x+y,L_1)}{2^{m(g-1,n+1)}} \right. \\
&\quad \left. + \sum_{a\in\mathcal{I}_{g,n}} \frac{V_{g_1,n_1+1}(x,L_{I_1})}{2^{m(g_1,n_1+1)}} \frac{V_{g_2,n_2+1}(y,L_{I_2})}{2^{m(g_2,n_2+1)}} H(x+y,L_1) \right) \mathrm{d}x\,\mathrm{d}y \\
&\quad + \frac{2^{-m(g,n-1)}}{2} \sum_{j=2}^n \int_0^\infty x\left(H(x,L_1+L_j)+H(x,L_1-L_j)\right) V_{g,n-1}(x,L_2,\dots,\widehat{L_j},\dots,L_n)\mathrm{d}x
\end{aligned} \tag{4.18}$$

Proof. Consider the McShane identity (4.11). Integrate it over the moduli space and take the partial derivative with respect to L_1. Then we have

$$\frac{\partial}{\partial L_1} L_1 V_{g,n}(L) = \frac{\partial}{\partial L_1}\int_{\mathcal{M}_{g,n}(L)} \tilde{\mathcal{D}}\ \mathrm{vol}_{WP} + \sum_{j=2}^n \frac{\partial}{\partial L_1}\int_{\mathcal{M}_{g,n}(L)} \tilde{\mathcal{R}}_j\ \mathrm{vol}_{\mathrm{WP}}.$$

Inserting Lemmas 4.31 and 4.32 we obtain the recursion relation (4.18). □

4.3 Calculation of the volumes

In this section we will supplement the recursion relation of Theorem 4.33 by initial conditions and show that it is indeed a recursion relation, i.e. that $V_{g,n}$ can be calculated from lower volumes. Furthermore we will see some examples and simplify the recursion relation in order to see that the volumes are actually polynomials in the length of the boundary components.

4.3.1 The case $g = n = 1$

Let us investigate the case $(g,n) = (1,1)$ a little bit further. Pick a good curve γ on $\Sigma_{1,1}$. Then the cut surface $\Sigma_{1,1}(\gamma) \simeq \Sigma_{0,3}$ is diffeomorphic to a pair of pants, two of which boundary curves have the same length. Since γ decomposes the surface we can use the associated Fenchel–Nielsen coordinates for the Teichmüller space $\mathcal{T}_{g,n}(L)$. This means

$$\mathcal{T}_{g,n}(L) \simeq \mathbb{R}_+ \times \mathbb{R},$$

where the coordinates are called (l,τ).
In order to integrate over $\mathcal{M}^\Gamma_{1,1}(L)$ we need to determine $\mathrm{Stab}(\gamma)$. The stabilizer of γ is given by isotopy classes of orientation-preserving diffeomorphisms fixing the boundary β and γ setwise. Since the Dehn twist ϕ_γ around γ fixes the curve γ we have $\phi_\gamma \in \mathrm{Stab}(\gamma)$. Therefore it is enough to look for orientation preserving diffeomorphisms of pairs of pants fixing two boundary components pointwise. By [13] we know that $\mathrm{Mod}_{1,1}$ is generated by a finite set of Dehn twists and possibly half twists. Since there are no more essential closed simple curves on the pair of pants the only possibility would be a Dehn twist around the boundary which is itotopic to the identity or a half twist interchanging two boundaries which is not allowed. Thus

$$\mathrm{Stab}(\gamma) = \{\phi_\gamma^n \mid n \in \mathbb{Z}\}.$$

In Fenchel–Nielsen coordinates this is the mapping $(l,\tau) \longmapsto (l, \tau + l)$. Thus we have

$$\mathcal{M}^{\Gamma}_{1,1}(L) \simeq \{(l,\tau) \mid l > 0, 0 \leq \tau \leq l\}\big/(x,0) \sim (x,x),$$

because $\{(l,\tau) \mid l > 0, 0 \leq \tau < l\}$ is a fundamental region for the action. In Fenchel–Nielsen coordinates we know $\omega_{\mathrm{WP}} = \mathrm{d}l \wedge \mathrm{d}\tau$ on Teichmüller space.

So we see that $\mathcal{M}^{\Gamma}_{1,1}(L)$ is symplectomorphic to the symplectic manifold $(\{(l,\tau) \mid l > 0, 0 \leq \tau \leq l\}\big/(x,0) \sim (x,x), \mathrm{d}l \wedge \mathrm{d}\tau)$. Now we can use this explicit coordinate system to perform the integration directly. Note that this space is in fact a manifold.

4.3.2 Initial conditions

We will need two intial conditions. The first one is

$$V_{0,3}(L) = 1 \tag{4.19}$$

for $L \in \mathbb{R}^3_+$. This is true since the hyperbolic structure of a pair of pants is unique up to self-homeomorphisms and thus $\mathcal{M}_{0,3} = \{\mathrm{pt}\}$. The second one is

Lemma 4.34. *Consider $\mathcal{M}_{1,1}(L)$ for $L \in \mathbb{R}_+$, i.e. the moduli space of 1-bordered tori. One has*

$$V_{1,1}(L) = \frac{\pi^2}{6} + \frac{L^2}{24}. \tag{4.20}$$

Proof. For $g = n = 1$ and $X \in \mathcal{T}_{1,1}(L)$ the McShane identity reads

$$\sum_{\gamma} \mathcal{D}(L, l_\gamma(X), l_\gamma(X)) = L,$$

because if $\gamma = \gamma_1 + \gamma_2$ with $\gamma_1 \neq \gamma_2$ all representatives of the free homotopy classes intersect on $\Sigma_{1,1}$ and thus cannot bound an embedded pair of pants. Therefore $\gamma_1 = \gamma_2$ and the sum ranges over all free homotopy classes of non-peripheral simple closed curves.

As in the calculation of the recursion relation we integrate this equation over the moduli space and take its L-derivative.

$$\begin{aligned}
\frac{\partial}{\partial L} L V_{1,1}(L) &= \frac{\partial}{\partial L} \int_{\mathcal{M}_{1,1}(L)} \sum_{\gamma} \mathcal{D}(L, l_\gamma(\cdot), l_\gamma(\cdot)) \, \mathrm{vol}_{\mathrm{WP}} \\
&= \frac{\partial}{\partial L} \int_{\mathcal{M}_{1,1}(L)} \pi^{\Gamma}_{*} \mathcal{D}(L, l_\gamma(\cdot), l_\gamma(\cdot)) \, \mathrm{vol}_{\mathrm{WP}} \\
&= \frac{\partial}{\partial L} \int_{\mathcal{M}^{\Gamma}_{1,1}(L)} \mathcal{D}(L, l(\cdot), l(\cdot)) \pi^{\Gamma *} \mathrm{vol}_{\mathrm{WP}} \\
&= \frac{\partial}{\partial L} \int_0^\infty \int_0^l \mathcal{D}(L, l, l) \, \mathrm{d}\tau \mathrm{d}l \\
&= \frac{\partial}{\partial L} \int_0^\infty x \mathcal{D}(L, x, x) \, \mathrm{d}x,
\end{aligned}$$

where we have used Definition 4.11 and Lemma 4.7. Now it remains to use (4.12) and integrate the expression

$$\begin{aligned}
\frac{\partial}{\partial L} L V_{1,1}(L) &= \int_0^\infty x \left(\frac{1}{1+e^{x+\frac{L}{2}}} + \frac{1}{1+e^{x-\frac{L}{2}}} \right) \mathrm{d}x \\
&= \int_{\frac{L}{2}}^\infty \frac{y-\frac{L}{2}}{1+e^y} \,\mathrm{d}y + \int_{-\frac{L}{2}}^\infty \frac{y+\frac{L}{2}}{1+e^y} \,\mathrm{d}y \\
&= 2\int_0^\infty \frac{y}{1+e^y} \,\mathrm{d}y - \int_0^{\frac{L}{2}} \frac{y}{1+e^y} \,\mathrm{d}y + \int_{-\frac{L}{2}}^0 \frac{y}{1+e^y} \,\mathrm{d}y + \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\frac{L}{2}}{1+e^y} \,\mathrm{d}y \\
&= \frac{\pi^2}{6} + \int_{-\frac{L}{2}}^0 \frac{y+\frac{L}{2}}{1+e^y} \,\mathrm{d}y - \int_0^{\frac{L}{2}} \frac{y-\frac{L}{2}}{1+e^y} \,\mathrm{d}y \\
&= \frac{\pi^2}{6} - \int_0^{\frac{L}{2}} \left(y - \frac{L}{2} \right) \left(\frac{1}{1+e^{-y}} + \frac{1}{1+e^y} \right) \mathrm{d}y \\
&= \frac{\pi^2}{6} + \frac{L^2}{8},
\end{aligned}$$

where we have used $\int_0^\infty \frac{y}{1+e^y} \mathrm{d}y = \frac{\pi^2}{12}$ without proof and $\left(\frac{1}{1+e^{-y}} + \frac{1}{1+e^y} \right) = 1$ which is easily checked. Thus,

$$V_{1,1}(L) = \frac{\pi^2}{6} + \frac{L^2}{24},$$

because possibly constant terms must vanish due to $LV_{1,1}(L) = 0$ for $L = 0$. □

Theorem 4.35. *The relation (4.18) together with the initial conditions $V_{0,3} = 1$ and $V_{1,1}(L) = \frac{\pi^2}{6} + \frac{L^2}{24}$ determine all Weil–Petersson volumes $V_{g,n}(L)$ recursively.*

Proof. Let us investigate on which volumes the recursion relation depends. For later reference let us denote the terms appearing in (4.18) as follows:

$$\begin{aligned}
\mathcal{A}_{g,n}^{\mathrm{con}}(L) &:= \frac{1}{2} \int_0^\infty \int_0^\infty xy \left(\frac{V_{g-1,n+1}(x, y, L_2, \ldots, L_n) H(x+y, L_1)}{2^{m(g-1,n+1)}} \right) \mathrm{d}x \,\mathrm{d}y, \\
\mathcal{A}_{g,n}^{\mathrm{dcon}}(L) &:= \frac{1}{2} \int_0^\infty \int_0^\infty xy \left(\sum_{a \in \mathcal{I}_{g,n}} \frac{V_{g_1,n_1+1}(x, L_{I_1})}{2^{m(g_1,n_1+1)}} \frac{V_{g_2,n_2+1}(y, L_{I_2})}{2^{m(g_2,n_2+1)}} H(x+y, L_1) \right) \mathrm{d}x \,\mathrm{d}y, \\
\mathcal{B}_{g,n}(L) &:= \frac{2^{-m(g,n-1)}}{2} \sum_{j=2}^n \int_0^\infty x \left(H(x, L_1 + L_j) + H(x, L_1 - L_j) \right) \\
&\qquad \times V_{g,n-1}(x, L_2, \ldots, \widehat{L_j}, \ldots, L_n) \,\mathrm{d}x,
\end{aligned}$$

such that the expression reads

$$\frac{\partial}{\partial L_1} L_1 V_{g,n}(L) = \mathcal{A}_{g,n}^{\mathrm{con}}(L) + \mathcal{A}_{g,n}^{\mathrm{dcon}}(L) + \mathcal{B}_{g,n}(L)$$

for $(g, n) \neq (1, 1)$ and $(0, 3)$. The term $\mathcal{A}_{g,n}^{\mathrm{con}}$ depends on $V_{g-1,n+1}$ and $\mathcal{B}_{g,n}$ on $V_{g,n-1}$. For $\mathcal{A}_{g,n}^{\mathrm{dcon}}$ it is slightly more complicated. We know $g_1 + g_2 = g$, $n_1 + n_2 = n - 1$ and $2 \leq 2g_i + n_i$ for $i = 1, 2$. Thus this term depends on $V_{i,j}$ with $i = g_1$ and $j = n_1$ without loss of generality and $2 \leq 2(g-g_1)+n-1$, i.e. $2g_1+n_1+g_1 \leq 2g+n-3+g-g_1$ and thus $3i+j \leq 3g+n-3-g_2 < 3g+n$. So we can compute the volumes inductively in $m = 3g + n$. However, as the recursion relation is only valid for $m \geq 4$ and $(g, n) \neq (1, 1)$ this argument only holds for $m \geq 5$. So it remains to see that we can calculate all volumes for $m \leq 4$. However, there is only $(1, 1), (0, 3)$ and $(0, 4)$. The first two we have already determined and the last one is computable via the recursion relation since it only depends on $V_{0,3}$ and since the relation is indeed valid for $(0,4)$. $\square$

4.3.3 Polynomial behaviour

In order to understand the structure of the recursion relations a little bit better we will investigate functions of the kind $\int_0^\infty \int_0^\infty x^k y^l H(x+y,t)\mathrm{d}x\mathrm{d}y$. In the following the natural numbers include zero, i.e. these statements all hold for indices in $\mathbb{N} = \{0,1,2,\ldots\}$.

Definition 4.36. For $i \in \mathbb{N}$, define $F_{2i+1} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by

$$F_{2i+1}(t) = \int_0^\infty x^{2i+1} H(x,t)\mathrm{d}x.$$

Proposition 4.37. *We have*

1. *for* $n, m \in \mathbb{N}$

$$\int_0^T y^m (T-y)^n \mathrm{d}y = \frac{m!n!}{(m+n+1)!} T^{m+n+1}, \tag{4.21}$$

2. *for* $i, j \in \mathbb{N}$

$$\int_0^\infty \int_0^\infty x^{2i+1} y^{2j+1} H(x+y,t)\mathrm{d}x\mathrm{d}y = \frac{(2i+1)!(2j+1)!}{(2j+2i+3)!} F_{2i+2j+3}(t), \tag{4.22}$$

3. *for* $k \in \mathbb{N}$

$$\frac{F_{2k+1}(t)}{(2k+1)!} = \sum_{i=0}^{k+1} \zeta(2i)\left(2^{2i+1}-4\right)\frac{t^{2k+2-2i}}{(2k+2-2i)!}. \tag{4.23}$$

Here, ζ is the Riemann ζ-function with values $\zeta(0) = -\frac{1}{2}$ and $\zeta(2) = \frac{\pi^2}{6}$.

Proof. The proof is a straight-forward calculation

1.

$$\begin{aligned}
\int_0^T y^m (T-y)^n \mathrm{d}y &= \int_0^T \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} T^k y^{n-k+m} \mathrm{d}y \\
&= \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} T^k \frac{1}{n+m-k+1} T^{n+m-k+1} \\
&= \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!(n-k)!} \frac{1}{n+m-k+1} T^{n+m+1} \\
&= \frac{n!m!}{(n+m+1)!} T^{n+m+1} \sum_{k=0}^{n} \frac{(-1)^{n-k}(m+n+1)!}{k!(n-k)!(n+m-k+1)m!} \\
&= \frac{n!m!}{(n+m+1)!} T^{n+m+1},
\end{aligned}$$

where we have used $\sum_{k=0}^n \frac{(-1)^{n-k}(m+n+1)!}{k!(n-k)!(n+m-k+1)m!} = 1$ without proof.

2.

$$\begin{aligned}\int_0^\infty \int_0^\infty x^{2i+1} y^{2j+1} H(x+y,t)\mathrm{d}x\mathrm{d}y &= \int_0^\infty \int_0^z (z-y)^{2i+1} y^{2j+1} \mathrm{d}y H(z,t)\mathrm{d}z \\ &= \frac{(2i+1)!(2j+1)!}{(2i+2j+3)!} \int_0^\infty z^{2i+2j+3} H(z,t)\mathrm{d}z \\ &= \frac{(2i+1)!(2j+1)!}{(2i+2j+3)!} F_{2i+2j+3}(t)\end{aligned}$$

3.

$$\begin{aligned}\frac{F_{2k+1}(t)}{(2k+1)!} &= \frac{1}{(2k+1)!}\int_0^\infty x^{2k+1}\left(\frac{1}{1+e^{\frac{x+t}{2}}} + \frac{1}{1+e^{\frac{x-t}{2}}}\right)\mathrm{d}x \\ &= \frac{1}{(2k+1)!}\left(2\int_0^\infty \frac{(2x+t)^{2k+1}+(2x-t)^{2k+1}}{1+e^x}\mathrm{d}x\right. \\ &\qquad\left. + \int_{-\frac{t}{2}}^0 \frac{2(2x+t)^{2k+1}}{1+e^x}\mathrm{d}x - \int_0^{\frac{t}{2}} \frac{2(2x-t)^{2k+1}}{1+e^x}\mathrm{d}x\right) \\ &= \frac{1}{(2k+1)!}\left(2\sum_{l=0}^{2k+1}\binom{2k+1}{l}\int_0^\infty \frac{(2x)^{2k+1-l}t^l + (2x)^{2k+1-l}t^l(-1)^l}{1+e^x}\mathrm{d}x\right. \\ &\qquad\left. -2\int_0^{\frac{t}{2}} (2x-t)^{2k+1}\left(\frac{1}{1+e^{-x}}+\frac{1}{1+e^x}\right)\mathrm{d}x\right) \\ &= \frac{2}{(2k+1)!}\sum_{l=0}^{2k+1}\binom{2k+1}{l} t^{2k+1-l}\left((-1)^{2k+1-l}+1\right)\int_0^\infty \frac{(2x)^l}{1+e^x}\mathrm{d}x \\ &\qquad - \frac{1}{(2k+1)!}\int_{-t}^0 y^{2k+1}\mathrm{d}y\end{aligned}$$

Evaluating the last integral and substituting $l = 2j+1$ because for even j the terms vanish we get

$$\begin{aligned}\frac{F_{2k+1}(t)}{(2k+1)!} &= \frac{t^{2k+2}}{(2k+2)!} + \frac{4}{(2k+1)!}\sum_{j=0}^{k}\binom{2k+1}{2j+1} 2^{2j+1} t^{2k-2j}\int_0^\infty \frac{x^{2j+1}}{1+e^x}\mathrm{d}x \\ &= \frac{t^{2k+2}}{(2k+2)!} + \frac{4}{(2k+1)!}\sum_{i=1}^{k+1}\binom{2k+1}{2i-1} t^{2k-2i+2} 2^{2i-1}\int_0^\infty \frac{x^{2i-1}}{1+e^x}\mathrm{d}x.\end{aligned}$$

Now we use the following lemma without proof, see [25].

Lemma 4.38. *We have*

$$\int_0^\infty \frac{x^{2i-1}}{1+e^x}dx = \frac{1}{2}\zeta(2i)(2i-1)!\left(2-2^{-2i+2}\right).$$

Therefore we obtain

$$\begin{aligned}\frac{F_{2k+1}(t)}{(2k+1)!} &= \frac{t^{2k+2}}{(2k+2)!} + \frac{1}{(2k+1)!}\sum_{i=1}^{k+1}\binom{2k+1}{2i-1}t^{2k-2i+2}2^{2i}\zeta(2i)(2i-1)!(2-2^{-2i+2})\\ &= \frac{t^{2k+2}}{(2k+2)!} + \sum_{i=1}^{k+1}\zeta(2i)(2^{2i+1}-4)\frac{t^{2k+2-2i}}{(2k+2-2i)!}\\ &= \sum_{i=0}^{k+1}\zeta(2i)\left(2^{2i+1}-4\right)\frac{t^{2k+2-2i}}{(2k+2-2i)!}.\end{aligned}$$

□

Now we use these calculations to show, first of all, that the Weil–Petersson volumes are polynomials in the L_i^2.

Proposition 4.39. *The Weil–Petersson volumes of moduli spaces of bordered Riemann surfaces are polynomials in the lengths of the boundary components, i.e.*

$$V_{g,n}(L) = \sum_{\substack{\alpha\in\mathbb{N}^n\\ |\alpha|\le 3g-3+n}} C_\alpha^{g,n} L^{2\alpha}, \tag{4.24}$$

where α is a n-multiindex of non-negative integers and $|\alpha| = \sum_{i=1}^n \alpha_i$.

Remark 4.40. We will see in the next chapter that it is indeed possible to see this result via application of the Duistermaat–Heckmaan result. The idea is that by adjoining hyperbolic pairs of pants with two cusps to the boundary geodesics we see that we can identify the space $\mathcal{M}_{g,n}(L)$ with some reduced space of $\mathcal{M}_{g,2n}$. This reduction is with respect to the Weil–Petersson symplectic form, the torus action obtained by rotating around the geodesics used for gluing and the length function squared as the momentum map. Thus formally one expects a polynomial behaviour in L^2. Details will follow in Section 5.1.3.

Proof. Since the volume $V_{g,n}(L)$ is determined by the $V_{i,j}(L')$ for $3i+j<3g+n$ we can do an induction over $3g+n$. The volumes $V_{0,3}$ and $V_{1,1}(L)$ are obviously polynomials in L^2 of degree 0 and 1, respectively.
Now suppose that the induction hypothesis holds for all $V_{i,j}(L)$ with $3i+j<3g+n$. Then we can compute $V_{g,n}$ via

$$\frac{\partial}{\partial L_1}L_1V_{g,n}(L) = \mathcal{A}_{g,n}^{\mathrm{con}}(L) + \mathcal{A}_{g,n}^{\mathrm{dcon}}(L) + \mathcal{B}_{g,n}(L).$$

Let us first look at the term $\mathcal{B}_{g,n}(L)$, i.e.

$$\frac{2^{-m(g,n-1)}}{2}\sum_{j=2}^{n}\int_0^\infty x\left(H(x,L_1+L_j)+H(x,L_1-L_j)\right)V_{g,n-1}(x,L_2,\dots,\widehat{L_j},\dots,L_n)\,\mathrm{d}x.$$

Since $3g+n-1<3g+n$ we have by induction hypothesis that $V_{g,n-1}(x,L_2,\dots,\widehat{L_j},\dots,L_n)$ is a polynomial in $x^2, L_2^2,\dots,L_n^2$ of order $3g-4+n$. Thus it is a sum of terms of the form (for $i\le 3g-4+n$)

$$\int_0^\infty x^{2i+1}\left(H(x,L_1+L_j)+H(x,L_1-L_j)\right)\mathrm{d}x$$

which is by Definition 4.36 equal to

$$F_{2i+1}(L_1+L_j)+F_{2i+1}(L_1-L_j).$$

By Proposition 4.37 this is a polynomial of order $2i+2$ in the components of L. Thus we have for $\mathcal{B}_{g,n}(L)$

$$2^{-m(g,n-1)-1)}\sum_{j=2}^{n}\sum_{i=0}^{3g-4+n}\sum_{\substack{\alpha\in\mathbb{N}^{n-2}\\|\alpha|\leq 3g-4+n-i}} C_{\alpha,i}^{g,n-1}(F_{2i+1}(L_1+L_j)+F_{2i+1}(L_1-L_j))L_2^{2\alpha_2}\cdots\widehat{L_j}\cdots L_n^{2\alpha_n},$$

which is a polynomial in the L_i^2 of order $3g-4+n-i+i+1=3g-3+n$. Here, the constants $C_{\alpha,i}^{g,n-1}$ are the constants we have by induction hypothesis for $V_{g,n-1}(x,L_2,\ldots,\widehat{L_j},\ldots,L_n)$ in front of $x^{2i}L_2^{2\alpha_2}\cdots\widehat{L_j}^{2\alpha_j}\cdots L_n^{2\alpha_n}$.

Now we consider the term

$$\mathcal{A}_{g,n}^{\mathrm{con}}(L)=\frac{1}{2}\int_0^\infty\int_0^\infty xy\left(\frac{V_{g-1,n+1}(x,y,L_2,\ldots,L_n)H(x+y,L_1)}{2^{m(g-1,n+1)}}\right)\mathrm{d}x\,\mathrm{d}y.$$

By induction hypothesis $V_{g-1,n+1}(x,y,L_2,\ldots,L_n)$ is a polynomial in $x^2,y^2,L_2^2,\ldots,L_n^2$ of order $3g-5+n$. Therefore the integral is a sum of terms like

$$\int_0^\infty\int_0^\infty x^{2i+1}y^{2j+1}H(x+y,L_1)\mathrm{d}x\mathrm{d}y=\frac{(2i+1)!(2j+1)!}{(2i+2j+3)!}F_{2i+2j+3}(L_1)$$

and thus $\mathcal{A}_{g,n}^{\mathrm{con}}(L)$ is

$$2^{-m(g-1,n+1)-1)}\sum_{i,j=0}^{3g-5+n}\sum_{\substack{\alpha\in\mathbb{N}^{n-1}\\|\alpha|\leq 3g-5+n-i-j}} C_{\alpha,i,j}^{g-1,n+1}\frac{(2i+1)!(2j+1)!}{(2i+2j+3)!}F_{2i+2j+3}(L_1)L_2^{2\alpha_2}\cdots L_n^{2\alpha_n}$$

which is a polynomial in the L_i^2 of order $3g-5+n-i-j+i+j+2=3g-3+n$.

The third term looks like

$$\mathcal{A}_{g,n}^{\mathrm{dcon}}(L)=\frac{1}{2}\int_0^\infty\int_0^\infty xy\left(\sum_{a\in\mathcal{I}_{g,n}}\frac{V_{g_1,n_1+1}(x,L_{I_1})}{2^{m(g_1,n_1+1)}}\frac{V_{g_2,n_2+1}(y,L_{I_2})}{2^{m(g_2,n_2+1)}}H(x+y,L_1)\right)\mathrm{d}x\,\mathrm{d}y,$$

where $V_{g_i,n_i+1}(x,L_{I_i})$ is a polynomial of order $3g_i-2+n_i$ in the x^2 and $L_{I_i}^2$. Thus the integral gives as in Proposition 4.37 $\frac{(2i+1)!(2j+1)!}{(2i+2j+3)!}F_{2i+2j+3}(L_1)$ and we have

$$\frac{1}{2}\sum_{a\in\mathcal{I}_{g,n}}\sum_{\substack{\alpha\in\mathbb{N}^{n_1+1},\beta\in\mathbb{N}^{n_2+1}\\|\alpha|\leq 3g_1-2+n_1\\|\beta|\leq 3g_2-2+n_2}}\frac{(2\alpha_1+1)!(2\beta_1+1)!}{(2\alpha_1+2\beta_1+3)!}\frac{C_\alpha^{g_1,n_1+1}}{2^{m(g_1,n_1+1)}}\frac{C_\beta^{g_2,n_2+1}}{2^{m(g_2,n_2+1)}}F_{2\alpha_1+2\beta_1+3}(L_1)L_{I_1}^{\alpha_{I_1}}L_{I_2}^{\beta_{I_2}},$$

which is a polynomial of degree $3g-3+n$. Thus the sum of all three terms is a polynomial of order $3g-3+n$ in the L_i^2. Integration over L_1 raises the power of L_1 by one which is again reduced by dividing by L_1. Thus $V_{g,n}$ is a polynomial of degree $3g-3+n$ in the L_i^2. □

4.3.4 Examples and special cases

In this subsection we want to look at one example and at some cases in which the recursion formula simplifies a lot.

Example 4.41. Let us use the recursion formula to calculate $V_{0,4}$, i.e. the first volume after the initial conditions.

The recursion relation reads

$$\frac{\partial}{\partial L_1} L_1 V_{0,4}(L) = \frac{1}{2}\sum_{j=2}^{4}(F_1(L_1+L_j)+F_1(L_1-L_j)),$$

because there is no pair of disjoint curves in different homotopy classes which bound an embedded pair of pants with a boundary curve such that the other part of the surface is connected. Furthermore there is no $a \in \mathcal{I}_{g,n}$ satisfying $g_1+g_2=g=0$ and $n_1+n_2=n-1=3$ such that $V_{g_1,n_1+1} \neq 0 \neq V_{g_2,n_2+1}$. Using Proposition 4.37 we obtain

$$\begin{aligned}\frac{\partial}{\partial L_1} L_1 V_{0,4}(L) &= \frac{1}{2}\sum_{j=2}^{4}\left(\frac{(L_1+L_j)^2}{2!}+4\zeta(2)+\frac{(L_1-L_j)^2}{2!}+4\zeta(2)\right)\\ &= \frac{1}{2}\left(3L_1^2+L_2^2+L_3^2+L_4^2+24\zeta(2)\right).\end{aligned}$$

Knowing that $\zeta(2)=\frac{\pi^2}{6}$ we obtain

$$V_{0,4}(L) = \frac{1}{2}\left(4\pi^2+L_1^2+L_2^2+L_3^2+L_4^2\right).$$

In the next chapter we will see that the coefficients $C_\alpha^{g,n}$ of $V_{g,n}(L)$ can in fact be related to intersection numbers on the moduli space. Thus we will define now the corresponding interesting term and look at some special cases.

Definition 4.42. Let the volume of the moduli space of bordered Riemann surfaces be given by $V_{g,n}(L)=\sum_\alpha C_\alpha^{g,n}L^{2\alpha}$. Then we define

$$\langle \alpha_1,\ldots,\alpha_n\rangle_g = 2^{-m(g,n)}C_\alpha^{g,n}\,2^{|\alpha|}\alpha!,$$

where $\alpha! := \prod_{i=1}^n \alpha_i!$.

Remark 4.43.
1. Although we have not seen yet why we introduced this notation we will call the expressions $\langle \alpha_1,\ldots,\alpha_n\rangle_g$ intersection numbers.
2. The definition is such that the total number of arguments in $\langle\ldots\rangle$ determines the n with respect to which the coefficients C are chosen.
3. It is possible to see that the terms are actually symmetric in their arguments. We will later see that they correspond to integrals of 2-forms of the power given by the arguments over the moduli space. Since 2-forms commute we have that $\langle \alpha_1,\ldots,\alpha_n\rangle_g$ is symmetric in the α_i.
4. The factor of $\frac{1}{2}$ for $(g,n)=(1,1)$ is conventional, however, as we will see later it is chosen such that many other such factors disappear among relations of the intersection numbers.

Proposition 4.44.
1. *For $n>0, g\geq 0$ and $|\alpha|=3g-3+n\geq 0$ we have*

$$\langle 1,\alpha_1,\ldots,\alpha_n\rangle_g = (2g-2+n)\langle \alpha_1,\ldots,\alpha_n\rangle_g, \tag{4.25}$$

called the string equation.

2. *For $n>0, g\geq 0$ and $|\alpha|=3g-2+n\geq 0$ we have*

$$\langle 0,\alpha_1,\ldots,\alpha_n\rangle_g = \sum_{\alpha_i\neq 0}\langle \alpha_1,\ldots,\alpha_i-1,\ldots,\alpha_n\rangle_g, \tag{4.26}$$

called the dilaton equation.

3. For $n \geq 3, g = 0$ with $|\alpha| = n-3$ we have

$$\langle \alpha_1, \dots, \alpha_n \rangle_0 = \frac{|\alpha|!}{\alpha!}.$$

Proof. 1. We need to compare the coefficient of $L_0^2 L_1^{2\alpha_1} \cdots L_n^{2\alpha_n}$ in the recursion relation for $V_{g,n+1}(L_0, L)$ with $L \in \mathbb{R}_+^n$. Using the calculations from the proof of Proposition 4.39 we see that $\mathcal{A}_{g,n+1}^{\text{con}}(L)$ looks like

$$2^{-m(g-1,n+2)-1} \sum_{\substack{\beta \in \mathbb{N}^{n+2} \\ |\beta| \leq 3g-4+n}} C_\beta^{g-1,n+2} \frac{(2\beta_x+1)!(2\beta_y+1)!}{(2\beta_x+2\beta_y+3)!} F_{2\beta_x+2\beta_y+3}(L_0) L_1^{2\beta_1} \cdots L_n^{2\beta_n}.$$

The term in front of $L_0^2 L_1^{2\alpha_1} \cdots L_n^{2\alpha_n}$ satisfies $\beta_i = \alpha_i$ for $i = 1, \dots, n$ and thus $|\beta| = \beta_x + \beta_y + \sum_{i=1}^n \alpha_i = \beta_x + \beta_y + 3g - 3 + n$. On the other hand $|\beta| \leq 3g - 4 + n$ and thus $\beta_x + \beta_y \leq -1$, which is a contradiction. Thus, $\mathcal{A}_{g,n+1}^{\text{con}}(L)$ does not contribute.

For the term $\mathcal{A}_{g,n+1}^{\text{dcon}}(L)$ we get

$$\sum_{a \in \mathcal{I}_{g,n+1}} 2^{-m(g_1,n_1+1)-m(g_2,n_2+1)-1} \sum_{\substack{\gamma \in \mathbb{N}^{n_1+1} \\ \beta \in \mathbb{N}^{n_2+1} \\ |\gamma| \leq 3g_1-2+n_1 \\ |\beta| \leq 3g_2-2+n_2}} \frac{(2\gamma_1+1)!(2\beta_1+1)!}{(2\gamma_1+2\beta_1+3)!} C_\gamma^{g_1,n_1+1} C_\beta^{g_2,n_2+1}$$
$$F_{2\gamma_0+2\beta_0+3}(L_0) L_{I_1}^{2\gamma_{I_1}} L_{I_2}^{\beta_{2I_2}}$$

For a fixed a we get $\beta_{I_1} = \alpha_{I_1}$ and $\gamma_{I_2} = \alpha_{I_2}$. Thus $|\beta| + |\gamma| = \beta_0 + \gamma_0 + \sum_{i=1}^n \alpha_i = \beta_0 + \gamma_0 + 3g - 3 + n$ and $|\beta| + |\gamma| \leq 3g + n_1 + n_2 - 4 = 3g + n - 4$ which gives again a contradiction. Here we have used that $n_1 + n_2 = n + 1 - 1$.

The only term contributing to the power in question is $\mathcal{B}_{g,n}(L)$. It is given by

$$2^{-m(g,n)-1} \sum_{j=1}^n \sum_{\substack{\beta \in \mathbb{N}^n \\ |\beta| \leq 3g-3+n}} C_\beta^{g,n} (F_{2\beta_0+1}(L_0 + L_j) + F_{2\beta_0+1}(L_0 - L_j)) L_1^{2\beta_1} \cdots \widehat{L_j} \cdots L_n^{2\beta_n}.$$

Since we are interested in terms proportional to $L_0^2 L_1^{2\alpha_1} \cdots L_n^{2\alpha_n}$ we get $\beta_i = \alpha_i$ for $i \neq j$ and $\beta_0 = \alpha_j$. So we need to look at the highest order term in $F_{2\alpha_j+1}$ since we need a factor of $L_j^{2\alpha_j} L_0^2$. Thus we have for the corresponding term

$$2^{-m(g,n)-1} \sum_{j=1}^n C_\alpha^{g,n} L_1^{2\alpha_1} \cdots \widehat{L_j} \cdots L_n^{2\alpha_n} \left(\frac{(L_0 + L_j)^{2\alpha_j+2}}{2\alpha_j+2} + \frac{(L_0 - L_j)^{2\alpha_j+2}}{2\alpha_j+2} \right)$$
$$\simeq 2^{-m(g,n)-1} \sum_{j=1}^n C_\alpha^{g,n} \binom{2\alpha_j+2}{2} \frac{2}{2\alpha_j+2} L_0^2 L_1^{2\alpha_1} \cdots L_n^{2\alpha_n}.$$

The prefactor is

$$2^{-m(g,n)} \sum_{j=1}^n C_\alpha^{g,n} \binom{2\alpha_j+2}{2} \frac{1}{2\alpha_j+2} = 2^{-m(g,n)} C_\alpha^{g,n} \sum_{j=1}^n \frac{(2\alpha_j+2)!}{2!(2\alpha_j)!(2\alpha_j+2)}$$
$$= 2^{-m(g,n)} C_\alpha^{g,n} \frac{3}{2}(2g-2+n) = \frac{3}{2}(2g-2+n) \frac{\langle \alpha_1, \dots, \alpha_n \rangle_g}{\alpha! 2^{3g-3+n}}$$

On the other side of the recursion relation the corresponding term is

$$\frac{\partial}{\partial L_0} L_0 V_{g,n+1}(L_0, L) \simeq C^{g,n+1}_{(1,\alpha)} 3 L_0^2 L_1^{2\alpha_1} \cdots L_n^{2\alpha_n}$$
$$= 3 \frac{\langle 1, \alpha_1, \ldots, \alpha_n \rangle_g}{\alpha! 2^{3g-3+n+1}} L_0^2 L_1^{2\alpha_1} \cdots L_n^{2\alpha_n}$$

and thus

$$\langle 1, \alpha_1, \ldots, \alpha_n \rangle_g = (2g - 2 + n) \langle \alpha_1, \ldots, \alpha_n \rangle_g.$$

2. The proof works in the same way as the proof of Proposition 4.39. We have to investigate the recursion relation for $V_{g,n+1}$ and look for the coefficient in front of $L_1^{2\alpha_1} \ldots L_n^{2\alpha_n}$. The same arguing as before shows that the contributions of $\mathcal{A}^{\mathrm{con}}_{g,n+1}(L_0, L)$ and $\mathcal{A}^{\mathrm{dcon}}_{g,n+1}(L_0, L)$ vanish. Thus we have

$$\frac{\langle 0, \alpha_1, \ldots, \alpha_n \rangle_g}{\alpha! 2^{3g-2+n}} L_1^{2\alpha_1} \cdots L_n^{2\alpha_n} = \frac{1}{2} \sum_{j=1}^{n} \left(\sum_{\substack{\beta \in \mathbb{N}^n \\ |\beta| \le 3g-3+n}} \frac{\langle \beta_x, \beta_1, \ldots, \widehat{\beta_j}, \ldots, \beta_n \rangle_g}{2^{|\beta|} \beta!} \right.$$
$$\left. (F_{2\beta_x+1}(L_0 + L_j) + F_{2\beta_x+1}(L_0 - L_j)) L_1^{2\beta_1} \cdots \widehat{L_j^{2\beta_j}} \cdots L_n^{2\beta_n} \right) [0, \alpha],$$

where $(\ldots)[0, \alpha]$ means the term with $L_1^{2\alpha_1} \cdots L_n^{2\alpha_n}$. Thus $\beta_i = \alpha_i$ for $i \neq 0, j$ and we see again that the condition on $|\beta|$ implies $\beta_x + |\alpha| - \alpha_j \le 3g - 3 + n$ and thus $\beta_x \le \alpha_j - 1$. Doing the same calculation with the F functions for slightly different powers as before we end up with

$$\frac{\langle 0, \alpha_1, \ldots, \alpha_n \rangle_g}{\alpha! 2^{3g-2+n}} = \frac{1}{2} \sum_{j=1}^{n} \frac{\langle \alpha_j - 1, \alpha_2, \ldots, \widehat{\alpha_j}, \ldots, \alpha_n \rangle_g \alpha_j!}{2^{|\alpha|-1} \alpha! 2\alpha_j (\alpha_j - 1)!} 2.$$

Collecting the terms and noticing that the $\langle \ldots \rangle_g$ are symmetric in the arguments as well as that they are zero for negative entries we obtain

$$\langle 0, \alpha_1, \ldots, \alpha_n \rangle_g = \sum_{j | \alpha_j \neq 0} \langle \alpha_1, \ldots, \alpha_j - 1, \ldots, \alpha_n \rangle_g$$

3. We will show this by induction on n. For $n = 3$ we have as the only possibility $\langle 0, 0, 0 \rangle_0 = 1$ which is true. Now suppose the statement holds for $n - 1$. Since $|\alpha| = n - 3 < n$ there must be at least one $\alpha_i = 0$. Assume without loss of generality that this is α_1. Then we have by (4.26)

$$\langle 0, \alpha_2, \ldots, \alpha_n \rangle_0 = \sum_{\alpha_i \neq 0} \langle \alpha_2, \ldots, \alpha_i - 1, \ldots, \alpha_n \rangle_0.$$

For the summands on the right hand side we have that the sum of their parameters is $|\alpha| - \alpha_i + \alpha_i - 1 = n - 4$ and there are $n - 1$ terms. Thus we can use the induction step and write

$$\begin{aligned}\langle \alpha_1, \ldots, \alpha_n \rangle_0 &= \sum_{\alpha_i \neq 0} \langle \alpha_2, \ldots, \alpha_i - 1, \ldots, \alpha_n \rangle_0 \\ &= \sum_{\alpha_i \neq 0} \frac{(n-4)!}{\alpha_2! \cdots (\alpha_i - 1)! \cdots \alpha_n!} \frac{\alpha_i}{\alpha_i} \\ &= \sum_{\alpha_i \neq 0} \frac{(|\alpha| - 1)! \alpha_i}{\alpha!} = \frac{|\alpha|!}{\alpha!},\end{aligned}$$

which is what we wanted to show. Thus we have calculated all intersection numbers for genus 0.

□

5 The Witten conjecture and two-dimensional gravity

5.1 The Witten conjecture

5.1.1 Deligne–Mumford compactification of the moduli space

In this first section we want to describe the so-called Deligne–Mumford compatification of the moduli space, denoted by $\overline{\mathcal{M}}_{g,n}$. It is a compactification of the moduli space by adjoining so-called nodal curves, i.e. we allow the surfaces to degenerate at discrete points, like in Figure 5.1a. Later we want to investigate the cohomology of the moduli space and this is of course simplified if the space is compact. The introduction will be based on [16]. A much more detailed exposition can be found in [30].

Definition 5.1. A Riemann surface Σ of genus g, n boundary curves and nodes $p_1, \dots, p_n$ is a compact connected Hausdorff space such that the following three conditions hold:

1. Every point $p \in \Sigma$ has a neighborhood either homeomorphic to the unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$, the half disk $\{z \in \mathbb{C} \mid |z| < 1, \mathrm{Im} z \geq 0\}$ or the set $\{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid z_1 z_2 = 0, |z_1| < 1, |z_2| < 1\}$ (called a node).

2. Every connected component (called a part) of $\Sigma \setminus \{p_1, \dots, p_m\}$ is a Riemann surface of genus $\bar{g}$ and $\bar{n}$ boundary components (including punctures) such that $2\bar{g} - 2 + \bar{n} > 0$.

3. If there are m nodes and k parts and the j-th part has genus g_j and n_j boundary components then $g = \sum_{j=1}^{k} g_j + m + 1 - k$ and $n = \sum_{j=1}^{k} n_j - 2m$.

Remark 5.2.

1. Such a surface is also called nodal curve.

2. Condition 1 implies that points are either smooth interior points, boundary points or so-called nodes, see Figure 5.1a. By compacteness there are only finitely many nodes.

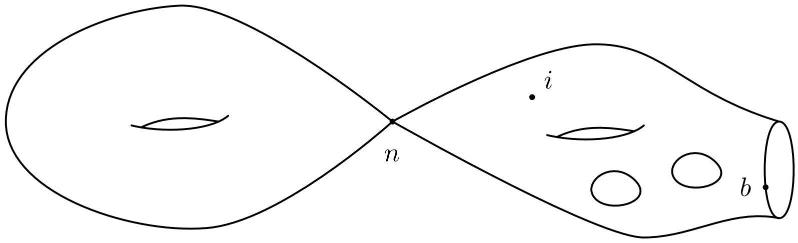
3. The second condition means that by removing the nodes one obtains a disconnected Riemann surface which has some genus, some boundaries and punctures from the removed nodal points and that it is stable, i.e. its automorphism group is finite. A curve with this condition is usually called stable curve, however here we only consider such surfaces and therefore we will use the term nodal or stable curve without distuingishing betweem them.

4. The third condition is to make sure that by opening all the nodes on the surface one obtains a Riemann surface of genus g and n boundaries, see Figure 5.1b.

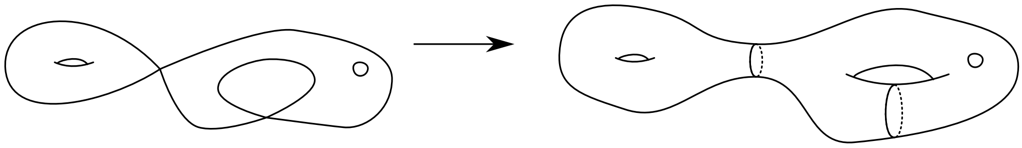
Next we define a biholomorphism between nodal curves such that we can add the biholomorphic equivalence class to the moduli space.

Definition 5.3. A homeomorphism $f : \Sigma \longrightarrow \Sigma'$ between two nodal curves is biholomorphic if it induces a biholomorphic mapping of the part Σ_j to a part of Σ' for every part Σ_j. If there exists a biholomorphic mapping between two nodal curves they are biholomorphically equivalent.

Remark 5.4. In a similiar manner we can transfer all considerations to nodal curves, i.e. for example that on each part Σ_j we can use the conformal class of metrics induced by $J_{|\Sigma_j}$ to determine a hyperbolic metric on this Riemann surface. Since the part Σ_j then has punctures we



(a)



(b)

Figure 5.1: Nodal curves

require that the hyperbolic metric is such that the nodes correspond to cusps. All in all we get a surface with finitely many nodes away from which we have a hyperbolic metric such that we can for example give the boundary components a length, since they cannot meet the nodes.

Definition 5.5. The Deligne–Mumford compactification of the moduli space $\overline{\mathcal{M}}_{g,n}(L)$ is the union of $\mathcal{M}_{g,n}(L)$ and all biholomorphic equivalence classes of nodal curves with at least one node such that the boundary curves have length L. As before, $\overline{\mathcal{M}}_{g,n}$ denotes the in this way compactified moduli space with punctures.

After having introduced $\overline{\mathcal{M}}_{g,n}(L)$ we need to give it now a topology and some more structure. Remembering Remark 5.2 we see that each nodal curve can be obtained by a Riemann surface of the appropriate genus and boundaries by degenerating certain closed simple non-peripheral curves to the length zero. This procedure of degeneration is of course not trivial, although the idea seems pretty clear. For a technical exposition we have to point to [16] and [30]. Remembering Remark 3.24 we see that the topology of $\mathcal{M}_{g,n}(L)$ can be stated in terms of Fenchel–Nielsen coordinates on the Teichmüller space. Furthermore since the subset topology induced on $\mathcal{M}_{g,n}(L)$ by the topology on the Deligne–Mumford space should coincide with the original one we will try to describe the definition of the topology in terms of Fenchel–Nielsen coordinates.
Let $[\Sigma] \in \overline{\mathcal{M}}_{g,n}(L)$ be obtained by an equivalence class of Riemann surfaces $[\Sigma_0] \in \mathcal{M}_{g,n}(L)$ by collapsing some curves $\{c_i\}_{i=1}^m$. One can see that there cannot be too many nodal points, as then some parts would have to be unstable. Thus it is possible to choose a set of decomposing curves for the Fenchel–Nielsen coordinates on Teichmüller space including the curves $\{c_i\}_{i=1}^m$. On Teichmüller space degenerating these curves means to allow the corresponding length variables l_{c_i} to vanish. By generalizing the Fenchel–Nielsen coordinates to nodal curves in the obvious way, the nodal curve Σ has coordinates in Teichmüller space $l_1, \dots, l_{3g-3+n}, \theta_1, \dots, \theta_{3g-3+n}$, of which m length coordinates l_{c_i} are zero. Then we define a set of fundamental neighborhoods of $[\Sigma] \in \overline{\mathcal{M}}_{g,n}(L)$ as the set of (δ, ϵ)-neighborhoods, where $\epsilon, \delta \in \mathbb{R}_+^{3g-3+n}$,

$$\begin{aligned} U_{(\epsilon,\delta)}([\Sigma]) := \{[\Sigma'] \in \overline{\mathcal{M}}_{g,n}(L) \mid |l_j(\Sigma) - l_j(\Sigma')| < \epsilon_j \;\forall\, j = 1, \dots, 3g-3+n, \\ |\theta_j(\Sigma) - \theta_j(\Sigma')| < \delta_j \;\forall\, j \text{ with } l_j(\Sigma) \neq 0\}. \end{aligned}$$

Theorem 5.6. *This topology on $\overline{\mathcal{M}}_{g,n}(L)$ is a compact Hausdorff topology.*

Proof. See [16] and [30]. □

Furthermore, the compactified moduli space is still an orbifold.

Theorem 5.7. *The Deligne–Mumford moduli space $\overline{\mathcal{M}}_{g,n}(L)$ is an orbifold and its compactification locus (i.e. $\overline{\mathcal{M}}_{g,n}(L) \setminus \mathcal{M}_{g,n}(L)$) is the transverse intersection of finitely many lower dimensional suborbifolds.*

Proof. See [16] and [30]. □

Since the compactification adds only lower-dimensional orbifolds we see that integration over the Deligne–Mumford compactification and over the usual moduli space are both well-defined and give the same results. However, we now have the advantage that $\overline{\mathcal{M}}_{g,n}(L)$ is compact and thus integrals of e.g. smooth forms always exist. One more thing that remains to be investigated is the Weil–Petersson form. By Wolperts theorem 3.30 we know that in Fenchel–Nielsen coordinates ω_{WP} is constant and thus extends smoothly to the boundary of $\mathcal{M}_{g,n}(L)$ in its compactification. Thus we obtain the Weil–Petersson symplectic form on $\overline{\mathcal{M}}_{g,n}(L)$.

5.1.2 Intersection numbers of tautological classes

In this chapter we want to define the necessary objects in order to see how we can use the Weil–Petersson volumes to determine the cohomology ring of the compactified moduli space of Riemann surfaces of genus g and with n punctures. Since the Deligne–Mumford moduli space is a compact orbifold we may consider its rational cohomology, i.e. $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. We will be interested in Chern classes of certain natural line bundles over the moduli space and the Weil–Petersson class. Their intersection product can then be used to determine parts of the cohomology ring. However, first we need to define these bundles and their Chern classes.

There are two kinds of bundles over the Deligne–Mumford orbifold that we are interested in. The first one is a complex line bundle over $\overline{\mathcal{M}}_{g,n}(L)$ whose fiber over each equivalence class of nodal curves is the cotangent space over one of the marked points. This complex line bundle has a Chern class which appears in many works on the cohomology of the moduli space. The other bundle is a S^1-bundle over the moduli space seen as the set of equivalence classes of hyperbolic surfaces such that we can associate to each such equivalence class one of the boundary components.

However, as the Deligne–Mumford orbifold does not arise as the orbit space of a group action on a manifold, it would be necessary to introduce these bundles in terms of orbit spaces of actions on groupoids which requires a totally different language. Thus we will not try to define these structures rigorously. But even on the non-compactified moduli space orbifold the notion of a vector bundle is slightly more delicate than on a usual manifold. This is why we will merely construct the bundles on Teichmüller space and then give a general argument why this gives a bundle on the orbifold and why it extends to the boundary of the compactification.

The complex line bundle $E_i \longrightarrow \mathcal{T}_{g,n}$

First, we define the complex line bundle over the space of all almost-complex structures on Riemann surfaces with punctures and then use the technique of [33] to show that it gives a bundle over Teichmüller space. Let Σ be a closed oriented Riemann surface of genus g.

Definition 5.8. 1. The set $\mathcal{A}_n$ is defined to be

$$\mathcal{A}_n := \{(J, z_1, \dots, z_n) \mid J \text{ almost complex structure on } \Sigma \text{ inducing the given orientation}, \\ z_1, \dots, z_n \in \Sigma \text{ pairwise different}\}.$$

2. The set $\tilde{E}_i$ for $i \in 1 \dots n$ is defined by

$$\tilde{E}_i = \{(J, z_1, \dots, z_n, v) \mid J \text{ almost complex structure on } \Sigma \text{ inducing the given orientation}, \\ z_1, \dots, z_n \in \Sigma \text{ pairwise different}, v \in \mathrm{T}_{z_i}\Sigma\}.$$

Theorem 5.9. *The projection $\tilde{\pi} : \tilde{E}_i \longrightarrow \mathcal{A}_n$ is a complex line bundle. It is Diff$_0$-equivariant and the trivialization descends to the quotient. Thus we obtain a complex line bundle $\pi : E_i \longrightarrow \mathcal{T}_{g,n}$.*

Remark 5.10. Since $\mathcal{A}_n$ is infinite-dimensional we have to say what kind of manifold it is. Unfortunately if we consider C^∞-structures it is not a Banach or Hilbert manifold and there is no general theorem telling us that $\mathcal{A}_n / \mathrm{Diff}_0 \simeq \mathcal{T}_{g,n}$ is a finite dimensional manifold. As was pointed out in Section 3.3 we actually need to consider some Sobolev $\mathcal{H}^s$-catgeory and then show that we obtain a smooth slice of C^∞-structures. Since this is technically quite difficult we will have to refer to the exact statements in [33].

Proof. In the sequel we will abbreviate the n-tupel of points by the letter z. Call the projection onto the Diff_0 quotients σ and $\tilde{\sigma}$, see next diagram.

$$\begin{tikzcd} \tilde{E}_i \arrow[r, "\tilde{\sigma}"] \arrow[d, "\tilde{\pi}"'] & \tilde{E}_i / \mathrm{Diff}_0 \arrow[d, "\pi"] \\ \mathcal{A}_n \arrow[r, "\sigma"] & \mathcal{A}_n / \mathrm{Diff}_0 \end{tikzcd} \tag{5.1}$$

Suppose we have a local trivialization of $\tilde{E}_i$, i.e. $\tilde{\Psi} : \tilde{\pi}^{-1}(U) \longrightarrow U \times \mathbb{C}$, where $U \subset \mathcal{A}_n$ is open. Then it may descend to the quotient, i.e. it gives a trivialization $\Psi : \pi^{-1}(\sigma(U)) \longrightarrow \sigma(U) \times \mathbb{C}$ defined by $x \longmapsto (\sigma(\tilde{\pi}(p)), \mathrm{pr}_2(\tilde{\Psi}(p)))$, where $p \in \tilde{E}_i$ such that $\tilde{\sigma}(p) = x$. However this is only well defined if for $g \cdot p$ one obtains the same elements, i.e.

$$(\sigma(\tilde{\pi}(g \cdot p)), \mathrm{pr}_2(\tilde{\Psi}(g \cdot p))) = (\sigma(\tilde{\pi}(p)), \mathrm{pr}_2(\tilde{\Psi}(p)))$$

for all $g \in \mathrm{Diff}_0$. This is equivalent to $\tilde{\pi}$ being Diff_0-equivariant and $\mathrm{pr}_2 \circ \tilde{\Psi}$ being Diff_0-invariant. Let us trivialize $\tilde{E}_i$ over some sufficiently small open neighborhood around $(\overline{J}, \overline{z}) \in \mathcal{A}_n$.

$$\begin{aligned} \tilde{\Psi} : \tilde{E}_i \,|_{\mathrm{Op}(\overline{J},\overline{z})} &\longrightarrow \mathrm{Op}(\overline{J},\overline{z}) \times \mathrm{T}_{\overline{z}_i}\Sigma \\ (J, z, v) &\longmapsto (J, z, \phi_{(J,z)}(v)), \end{aligned}$$

where $\phi_{(J,z)} : \mathrm{T}_{z_i}\Sigma \longrightarrow \mathrm{T}_{\overline{z}_i}\Sigma$ is an isomorphism of complex vector spaces with respect to J_{z_i} and $\overline{J}_{\overline{z}_i}$ and depends smoothly on the pair (J, z). Now we will make an assumption on ϕ.

Assumption 5.11. *For each* $(\overline{J}, \overline{z}) \in \mathcal{A}_n$ *there exists a family of isomorphisms of complex vector spaces*

$$\phi_{(J,z)} : (T_{z_i}\Sigma, J_{z_i}) \longrightarrow (T_{\overline{z}_i}\Sigma, \overline{J}_{\overline{z}_i})$$

depending smoothly on (J, z) *and satisfying*

$$\phi_{(g_* J, g(z))}(T_{z_i} g \cdot v) = \phi_{(J,z)}(v).$$

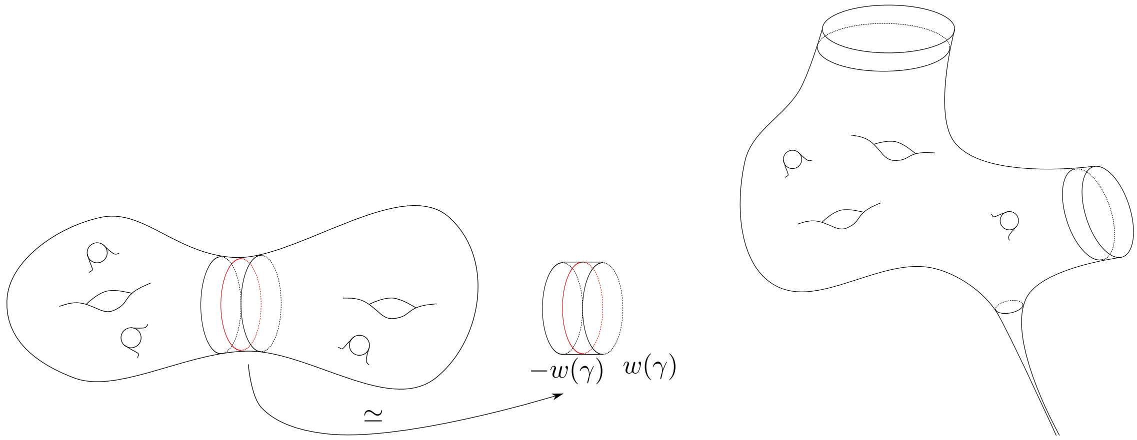
Now we see

$$\tilde{\pi}(g \cdot (J, z, v)) = \tilde{\pi}(g_* J, g(z), \mathrm{T}_{z_i} g \cdot v) = (g_* J, g(z)) = g \cdot \tilde{\pi}(J, z, v)$$

which means that $\tilde{\pi}$ descends to the projection $\pi : \tilde{E}_i / \mathrm{Diff}_0 \longrightarrow \mathcal{A}_n / \mathrm{Diff}_0$ and Diag. 5.1 commutes.
Furthermore we have

$$\mathrm{pr}_2 \circ \tilde{\Psi}(g \cdot (J, z, v)) = \phi_{(g_* J, g(z))}(\mathrm{T}_{z_i} g \cdot v) = \phi_z(v) = \mathrm{pr}_2 \circ \tilde{\Psi}(J, z, v)$$

and therefore that the trivialization descends as a map to the quotient.
Now we use the results of [33] to see that there exists a smooth slice in $\mathcal{A}_n$ bijective to $\mathcal{A}_n / \mathrm{Diff}_0$. This is the point where the analysis enters. Thus $\mathcal{A}_n / \mathrm{Diff}_0$ has a topology such that σ is continuous as well as a differentiable structure such that σ is smooth. Since Diag. 5.1 commutes we can use this to see that the trivializations Ψ are continuous. More over the trivializations combined with



(a) The collar neighborhood around a good curve (red)

(b) Collar neighborhoods around boundary curves and cusps

Figure 5.2

coordinates on $\mathcal{A}_n \diagup \mathrm{Diff}_0$ give local coordinates (as the fibre is already a vector space) and we see that $\tilde{E}_i \diagup \mathrm{Diff}_0$ is a differentiable manifold. It is constructed such that all the projections are smooth. Furthermore, by $\mathcal{A}_n \diagup \mathrm{Diff}_0 \simeq \mathcal{T}_{g,n}$ we obtain a complex line bundle $E_i \longrightarrow \mathcal{T}_{g,n}$. □

The S^1 principle bundles $P_i \longrightarrow \mathcal{T}_{g,n}(L)$ and $P_i' \longrightarrow \mathcal{T}_{g,n}$

Now we want to describe the construction of certain S^1-bundles over the Teichmüller space of Riemann surfaces with boundary. We want to define them in such a way that we can consider the limit $L \longrightarrow 0$ as well, i.e. if the boundaries become cusps. The idea is to mark a point on the i-th boundary such that we have a S^1-action on this point. Now, if the boundary becomes a cusp we have the problem that there is no circle anymore, thus in this case we would like to take a horocycle close to the cusp. Therefore we will in general not consider a point on the boundary but on a curve parallel and close to the boundary.

Theorem 5.12 (Collar theorem)**.** *Let X be a closed Riemann surface with a hyperbolic structure. Then, around each simple closed geodesic γ there exists a collar $\{p \in X \mid dist(p,\gamma) \leq w(\gamma) :=$ $\operatorname{arcsinh}\left((\sinh(l(\gamma)/2))^{-1}\right)$ isometric to $[-w(\gamma), w(\gamma)] \times S^1$ with metric $d\rho^2 + l^2(\gamma)\cosh(\rho^2)dt^2$, where t is the angular variable and ρ the length variable of the cylinder. Moreover, if two such geodesics are disjoint then the collar neighborhoods are also disjoint. See Figure 5.2a for a picture.*

Remark 5.13. By doubling a surface with boundary we see that each boundary geodesic has a half collar as a neighborhood and these collars are all disjoint. Furthermore this result can be extended to the case with cusps. There, the collar neighborhoods around simple closed geodesics do not intersect the cusps.

Proof. See [4] and [25]. □

Now we choose a function $F : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with the property that for each boundary component β_i of X there is a curve of constant curvature of length $F(l(\beta_i))$ inside the collar neighborhood of β_i and $\lim_{x\to 0} F(x) = \frac{1}{4}$.

Definition 5.14. On a hyperbolic surface X with n boundary components $\{\beta_i\}_{i=1}^n$ define the curve $\tilde{\beta}_i$ to be this curve of constant curvature and of length $F(l(\beta_i))$ inside the collar neighborhood of β_i. If X has cusps then we define the curve $\tilde{\beta}_i$ to be the curve of constant curvature and of length $\frac{1}{4}$ in the cusp neighborhood. Furthermore, we orient the curves in the following way. At $x \in \tilde{\beta}_i$ pick a perpendicular vector N_x pointing outwards (i.e. towards the cusp or the boundary) and define $v_x \in \mathrm{T}_x\tilde{\beta}_i$ to be positive if and only if (v_x, N_x) has positive orientation with respect to the orientation of X.

Remark 5.15. The idea is that this way we obtain a closed non-trivial curve around each boundary component or around each cusp, respectively. If the curve is around a boundary component then we have a canonical bijection between β_i and $\tilde{\beta}_i$ by the collar neighborhood theorem 5.12. Furthermore, by this theorem we see that all such curves are pairwise disjoint, see Figure 5.2b.

Let Σ be a fixed Riemann surface of genus g with n boundary components and $L \in \mathbb{R}^n_+$.

Definition 5.16. We define the following two sets:

$$\begin{aligned} \tilde{P}_i &:= \{(g,p) \mid g \text{ hyperbolic metric on } \Sigma \text{ such that the boundary components are geodesics} \\ &\qquad \text{of length } L, p \in \tilde{\beta}_j\} \\ \mathcal{M}_{-1} &:= \{g \mid g \text{ hyperbolic metric on } \Sigma \text{ such that the boundary components} \\ &\qquad \text{are geodesics of length } L\} \end{aligned}$$

Theorem 5.17. *The map $\tilde{P}_i \longrightarrow \mathcal{M}_{-1}$ is a S^1-principal bundle. The group Diff_0 acts on both spaces and the trivializations of the bundle are equivariant. One obtains as the quotient a bundle $\overline{P}_i \longrightarrow \mathcal{M}_{-1}/\mathit{Diff}_0$ which is isomorphic to a S^1-bundle $P_i \longrightarrow \mathcal{T}_{g,n}(L)$.*

Remark 5.18. Once more we ignore the technical difficulties arising from the fact that $\mathcal{M}_{-1}$ is infinite-dimensional.

Proof. The proof is in principle very close to the proof of Theorem 5.9. As we have seen in this proof we only need to find suitable trivializations and use [33] in order to get a manifold structure on the quotient. The reason for the latter is that once more the quotient of the base space is in fact the Teichmüller space. The group Diff_0 acts on $\tilde{\pi}_i$ by $(g,p) \longmapsto (\psi_* g, \psi(p))$ for $\psi \in \mathrm{Diff}_0$. Since ψ is an isometry of two hyperbolic surfaces it maps the horocylce β_i to $\psi(\beta_i)$ which is again a horocycle with respect to the metric $\psi_* g$, denote the dependence on the metric by a superscript g. So the trivialization needs to identify these circles in a suitable way. Let $\tilde{\Psi} : \tilde{\pi}^{-1}(\mathrm{Op}(\overline{g})) \longrightarrow \mathrm{Op}(\overline{g}) \times \tilde{\beta}_i^{\overline{g}}$ be a trivialization. As Diff_0 acts on the metric in the same way on both spaces the necessery condition for the projection to descend to the quotient is again immediate. However, for the invariance of the trivialization section we need the following assumption

Assumption 5.19. *For each such hyperbolic metric $\overline{g} \in \mathcal{M}'_{-1}$ there exists a family of maps*

$$\phi_g : \tilde{\beta}_i^g \longrightarrow \tilde{\beta}_i^{\overline{g}}$$

which is smooth in g, S^1-equivariant with respect to the length-normalized rotation and satisfies

$$\phi_{\psi_* g}(\psi(p)) = \phi_g(p).$$

This gives as in the proof of Theorem 5.9 trivializations on the quotient as maps. Then one proceeds as before by recalling that the quotient of the base is in fact the Teichmüller space and thus one introduces coordinates on the bundle as above to see that it is a manifold and that the projections are smooth. Thus, using again the diffeomorphism with Teichmüller space we obtain a S^1-principal bundle $P_i \longrightarrow \mathcal{T}_{g,n}(L)$. $\square$

Now it remains to construct this kind of bundle for the case of hyperbolic surfaces with cusps. Let Σ now be a Riemann surface of genus g without boundary.

Definition 5.20. We define

$$\begin{aligned}\tilde{P}_i' &:= \{(g, z_1, \ldots, z_n, p) \mid z_1, \ldots, z_n \in \Sigma \text{ pairwise different}, g \text{ hyperbolic metric on } \Sigma \\ &\qquad \text{with cusps in } z_1, \ldots, z_n, p \in \tilde{\beta}_i\} \\ \mathcal{M}'_{-1} &:= \{(g, z_1, \ldots, z_n) \mid z_1, \ldots, z_n \in \Sigma \text{ pairwise different}, g \text{ hyperbolic metric on } \Sigma \\ &\qquad \text{with cusps in } z_1, \ldots, z_n\}.\end{aligned}$$

Theorem 5.21. *The map $\tilde{P}_i' \longrightarrow \mathcal{M}'_{-1}$ defines a S^1-principal bundle with Diff$_0$-equivariant trivializations. We obtain as the quotient a S^1-principal bundle $\overline{P}_i' \longrightarrow \mathcal{M}'_{-1} \diagup \mathrm{Diff}_0$ which is isomorphic to the S^1-bundle $P_i' \longrightarrow \mathcal{T}_{g,n}$ given by*

$$P_i' = \{(X, p) \mid X \in \mathcal{T}_{g,n}, p \in \tilde{\beta}_i\}.$$

Proof. The proof works as before for Thms. 5.9 and 5.17. One has again the assumption 5.19 for the horocycle around the cusps. □

Bundle isomorphism, bundles over $\overline{\mathcal{M}}_{g,n}$ and tautological classes

Theorem 5.22. *The circle bundle in $E_i \longrightarrow \mathcal{T}_{g,n}$, denoted by $\overline{E}_i$, is isomorphic to $P_i' \longrightarrow \mathcal{T}_{g,n}$. This is orientation-reversing with respect to the canonical orientation on the complex line bundle and the chosen orienation for $\tilde{\beta}_i$.*

Proof. We will construct an isomorphism of the circle bundle in $\tilde{E}_i$ and the bundle $\tilde{P}_i'$ and then show that it descends to a smooth S^1-bundle isomorphism on the quotient bundles.
Consider the point $(J, z, v) \in \tilde{E}_i$. We restrict the fiber to $\mathrm{T}_z\Sigma \setminus \{0\}$ and identify the rays $v \sim \lambda v$ for $\lambda \in \mathbb{R}_+$. This is a circle bundle over $\mathcal{A}_n$ if we would choose smoothly a unique representative for each class. By [4] we know that some cusp region around z_i for the metric g_J is isometric to the punctured disc $\{z \in \mathbb{C}, z \neq 0, |z| \leq 1$. The geodesics are the rays perpendicular to the boundary circle and going to 0 which corresponds to the cusp. The horocycle $\tilde{\beta}_i$ corresponds to the circle around 0 with circumference $\frac{1}{4}$. Now take any point $p \in \tilde{\beta}_i$ Then there is a unique geodesic $\gamma_p(t)$ perpendicular to $\tilde{\beta}_i$ and going to the cusp. Consider its tangent vectors $\dot{\gamma}_p(t)$. Since the point 0 corresponds to a cusp their length (measured in the standard $\mathbb{C}$-metric) converges to zero when the curve approaches the cusp. Take any function λ increasing fast enough such that the length of $\lambda(t)\dot{\gamma}_p(t)$ converges to some vector $v_p \in \mathrm{T}_0\mathbb{C} \simeq \mathrm{T}_{z_i}\Sigma$ unequal to zero. So we get a map $p \longmapsto [\lim_{t\longrightarrow\infty} \lambda(t)\gamma_p(t)] \in \mathrm{T}_{z_i}\Sigma \setminus \{0\} \diagup \mathbb{R}_+$. This is independent of λ as another choice would multiply the resulting vector by some constant but not change its direction. It is smooth and bijective as can be seen in the local model $\mathbb{C} \setminus \{0\}$. Furhtermore if $\tilde{\beta}_i$ is parametrized proportional to arclength then the S^1-action corresponds to the constant rotation around the circle of radius $\frac{1}{4}$ in $\mathbb{C} \setminus \{0\}$. Then one sees that the map is in fact S^1-equivariant. This gives the following S^1-equivariant bundle map

$$\begin{aligned}\tilde{\overline{E}}_i &\longrightarrow \tilde{P}_i' \\ (J, z, [v_p]) &\longmapsto (g_J, z, p),\end{aligned}$$

where g_J is the unique hyperbolic metric in the conformal class defined by J and $\tilde{\overline{E}}_i$ denotes the circle bundle of $\tilde{E}_i$. The map $[v_p] \longmapsto p$ is the inverse of the map constructed above.
Before investigating whether the map descends to the Diff$_0$-quotient let us look at the orientation. Remember that we defined the orientation of $\tilde{\beta}_i$ such that $v_p \in \mathrm{T}_p\tilde{\beta}_i$ is positive if and only if (N_x, v_x) is positively oriented with respect to the orientation of Σ for some outward pointing vector. On a boundary this would be the induced orientation. However, as the cusp is inside the curve $\tilde{\beta}_i$ this is the opposite orientation as the one in $\mathrm{T}_{z_i}\Sigma$ and thus the map is orientation reversing, see Figure 5.3b.

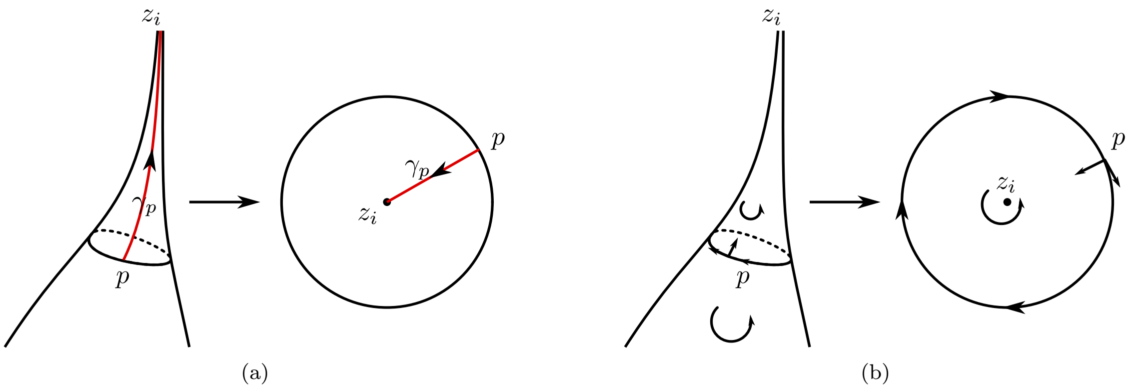


Figure 5.3: The local model for a cusp region. The closed curve is the horocycle of length $\frac{1}{4}$, $\tilde{\beta}_i$ and the red curve denotes a geodesic going up the cusp which is perpendicular to the horocylce

It is well defined on the quotient if

$$(g_{\phi_* J}, \phi(z), [\mathrm{T}_{z_i}\phi \cdot v_p]) \sim (g_J, z, \phi(p))$$

on P_i' for all $\phi \in \mathrm{Diff}_0$. By [33] we have $\phi_* g_J = g_{\phi_* J}$ and for the marked points this is obvious. Denote by $\Sigma^{g_J}_{\mathrm{Op}(z_i)}$ the hyperbolic cusp region of z_i until the curve $\tilde{\beta}_i$. Since we need to show that $\mathrm{T}_{z_i}\phi \cdot v_p$ corresponds to $\phi(p)$ under the map above we use the fact that the cusp neighborhoods are isometric to the disc of circumference $\frac{1}{4}$. The definition of the curves $\tilde{\beta}_i$ was such that they were horocylces of the length $\frac{1}{4}$ around the cusp. This means that the isometry ϕ between (Σ, g_J) and $(\Sigma, g_{\phi_* J})$ restricts to an isometry of the cusp neighborhoods $\Sigma^{g_J}_{\mathrm{Op}(z_i)} \simeq \Sigma^{g_{\phi_* J}}_{\mathrm{Op}(\phi(z_i))}$ as all isomorphism classes of cusp neighborhoods are determined by the length of the horocycle. Thus we obtain an isometry from the disc to itself mapping the image of v_p on the disc to the image of $\mathrm{T}_{z_i}\phi \cdot v_p$ on the disc. Since isometries map geodesics to geodesics we see that the geodesic relating v_p and p is mapped to the corresponding one through $\mathrm{T}_{z_i}\phi \cdot v_p$ and $\phi(p)$. Thus ϕ maps p to $\phi(p)$ and therefore the bundle map descends to the Diff_0-quotient $\overline{E_i} \longrightarrow P_i'$. Since the map was an orientation reversing isomorphism on the bundles and since it covers the identity (as $\mathcal{A}_n / \mathrm{Diff}_0 \simeq \mathcal{T}_{g,n} \simeq \mathcal{M}'_{-1} / \mathrm{Diff}_0$) we obtain a S^1-principal bundle isomorphism. $\square$

Now we have constructed two isomorphic bundles over Teichmüller space. The next step is to show that they give indeed isomorphic bundles in the orbifold sense on $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}(L)$, respectively.

Lemma 5.23. *The trivializations of the bundles $\overline{E}_i$, P_i and P_i' are $\mathrm{Mod}_{g,n}$-equivariant. All stabilizers are finite and the trivializations $V \times S^1$ are such that the kernel of the $\mathrm{Mod}_{g,n}$-action on $V \times S^1$ is equal to the kernel of the $\mathrm{Mod}_{g,n}$-action on $\mathcal{T}_{g,n}$ or $\mathcal{T}_{g,n}(L)$, respectively.*

Proof. In order to see that the trivializations descend to the moduli space we need the same arguments as before in the proofs of Thms. 5.9 and 5.17 but this time with respect to the Mod-action. For the projections it is again clear. However, for the trivialization sections we need Assumption 5.19 to hold for any diffeomorphism g, not just for $g \in \mathrm{Diff}_0$. Choose such a trivialization and we obtain bundles on the quotient.
The stabilizers are finite because they are the stabilizers of the Mod-action on the Teichmüller space which are known to be finite.
Since $\mathrm{Mod}_{g,n}$ acts effectively on Teichmüller space the kernels of the $\mathrm{Mod}_{g,n}$-action are all trivial. This means that the orbifold is in fact good, i.e. we obtain indeed a good complex orbifold vectorbundle over moduli space, see [31]. $\square$

Corollary 5.24. *By Lemma 5.23 and [31] the quotient bundles by $\mathrm{Mod}_{g,n}$ give rise to good orbifold bundles over the uncompactified moduli space $\mathcal{M}_{g,n}$ and $\mathcal{M}_{g,n}(L)$, respectively.*

Remark 5.25.
1. Since each part of a nodal curve must be stable we see that the marked points and the curves $\tilde{\beta}_i$ must be disjoint from the nodal points. Thus the bundles E_i, P_i and P_i' extend to $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}(L)$, respectively. Furthermore, the bundle isomorphisms can also be extended such that the extended bundles are still isomorphic.
2. We have seen that we obtain an orientation reversing bundle isomorphism from $\overline{E}_i \longrightarrow P_i'$. Dualizing the bundle E_i we obtain a complex line bundle over $\mathcal{M}_{g,n}$ where each fibre is the cotangent space of the marked point $\mathrm{T}_{z_i}\Sigma_g$. We call this bundle $\mathcal{L}_i'$ and the S^1-subbundle $\mathcal{L}_i$. The isomorphism thus gives us an orientation preserving S^1-bundle isomorphism $\mathcal{L}_i \longrightarrow P_i'$.

Corollary 5.26. *The Chern classes of the bundles $\mathcal{L}_i$ and P_i' agree, i.e.*

$$c_1(\mathcal{L}_i) = c_1(P_i') \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

Definition 5.27. Define the tautological classes $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ by

$$\psi_i := c_1(\mathcal{L}_i).$$

5.1.3 Connection to the Weil–Petersson volume

Having introduced the Chern classes of the bundles we now want to show that their intersection pairing can be calculated by calculating the Weil–Petersson volume of the moduli space. The idea is that the Weil–Petersson symplectic forms on the moduli spaces of length L can be seen as symplectic quotients with respect to a T^n-action on a bigger space and the length function as the moment map. This way one can use a standard trick to write the volume with respect to the symplectic forms in terms of an integral over curvature forms which describe the dependence of the symplectic form on a parameter, i.e. the length of the boundaries. In this section we will work this out, however, it is once more necessary to cite a couple of facts as there are still many technical difficulties.

Definition 5.28. Define the moduli space of bordered Riemann surfaces with marked points by

$$\begin{aligned}\widehat{\mathcal{M}}_{g,n} = &\{(X, p_1, \dots, p_n) \mid X \text{ a hyperbolic nodal curve of genus } g \text{ and } n \text{ boundary components} \\ &\beta_1, \dots, \beta_n, p_i \in \tilde{\beta}_i \forall i = 1, \dots, n\} \big/ \sim,\end{aligned}$$

where the equivalence relation is defined as $(X, p) \sim (Y, q) \iff \exists h : X \longrightarrow Y, h(p_i) = q_i$ for $i = 1, \dots, n$ and h an isometry of nodal curves.

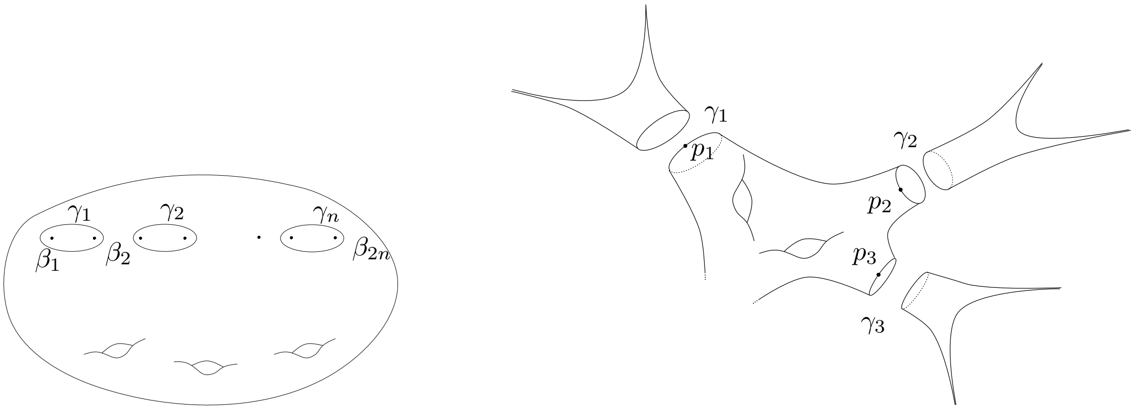
Remark 5.29.
1. Note that we did not specify the length of the boundary curves. This means that if we forget about the marked points on the boundary this is the union of $\overline{\mathcal{M}}_{g,n}(L)$ over all $L > 0$.
2. Similarly to the proofs in Section 5.1.2 one can show that $\widehat{\mathcal{M}}_{g,n}$ is again an orbifold of real dimension $6g - 6 + 4n$. The extra $2n$ dimensions come from the choice of the length of the boundary component and the position of the marked points.
3. In the sequel we will not show explicitly that the defined actions and maps are well defined on this space because we technically define them on the covering space without the quotient. However, this is not very difficult and is very close to the proofs we had at the end of Section 4.1.1.

Now we equip this space with a T^n-action, a symplectic structure coming from the Weil–Petersson symplectic structure on $\mathcal{M}_{g,n}(L)$ and a moment map $\frac{l^2}{2} : \widehat{\mathcal{M}}_{g,n} \longrightarrow \mathbb{R}_+^n$.

Definition 5.30. On $\widehat{\mathcal{M}}_{g,n}$ we define

1. a function $l : \widehat{\mathcal{M}}_{g,n} \longrightarrow \mathbb{R}_+^n$ by

$$l(X, p_1, \dots, p_n) = (l_{\beta_1}(X), \dots, l_{\beta_n}(X)),$$



(a) The marked points $\beta_1, \ldots, \beta_n$ and the chosen curves $\gamma_1, \ldots, \gamma_n$

(b) The gluing of suitable pairs of pants to the boundary. The twisting is fixed by choosing a geodesic perpendicular to the boundary and going up a cusp on each pair of pants and gluing it to the marked point on the boundary

Figure 5.4

2. a T^n-action, where $t = (t_1, \ldots, t_n) \in T^n$ acts via

$$t \cdot (X, p_1, \ldots, p_n) = \left(X, \xi_1^{t_1}(p_1), \ldots, \xi_n^{t_n}(p_n)\right),$$

where ξ_i^t is the rotation in the positive direction proportional to arclength around $\tilde{\beta}_i$. This means, if $\gamma_i : [0, b_i] \longrightarrow \tilde{\beta}_i$ is a parametrization proportional to arclength, then $\xi_i^t(\gamma_i(s)) := \gamma_i(s+tb_i)$ such that $\xi_i^t = \xi_i^{t+1}$. Thus we have a standard torus $(\mathbb{R}/\mathbb{Z})^n$ acting on the marked points.

Theorem 5.31. *On $\widehat{\mathcal{M}}_{g,n}$ there exists a symplectic structure $\widehat{\omega}$ such that*

1. *$\widehat{\omega}$ is T^n invariant,*
2. *$\frac{l^2}{2}$ is the moment map of the T^n-action and*
3. *the canonical map $s : l^{-1}(L)/T^n \longrightarrow \overline{\mathcal{M}}_{g,n}(L)$ is a symplectomorphism.*

Here, $\frac{l^2}{2} := \frac{1}{2}(l_1^2, \ldots, l_n^2)$ for $l = (l_1, \ldots, l_n)$.

Proof. First, we consider a surface $\Sigma_{g,2n}$ with genus g and $2n$ marked points $\beta_1, \ldots, \beta_{2n}$. Now we choose n closed simple disjoint curves on $\Sigma_{g,2n}$ such that for each $i = 1, \ldots, n$ γ_i bounds a pair of pants with β_{2i-1} and β_{2i}, see Figure 5.4a. Now define $\Gamma = (\gamma_1, \ldots, \gamma_n)$ as usual. Furthermore we define $\overline{\mathcal{M}}_{g,2n}^{\Gamma}$ as in Sect. 4.1.1 except that we now allow the curves to have nodes away from the chosen geodesics. Concretely this means

$$\overline{\mathcal{M}}_{g,2n}^{\Gamma} := \{(X, \eta) \mid X \text{ a hyperbolic nodal curve with genus } g \text{ and } 2n \text{ cusps}, \eta \in \mathcal{O}_\Gamma\}/\sim,$$

where $\mathcal{O}_\Gamma$ is defined as in Sect. 4.1.1 as the n-tuple of homotopy classes of elements of $\mathrm{Mod}_{g,n} \cdot \Gamma$. The equivalence relation is again $(X, \eta) \sim (Y, \rho) \iff \exists h : X \longrightarrow Y, h_*\eta = \rho$ with h an isometry of nodal curves. Note, that the hyperbolic surface X has nodes and cusps, but no boundary components. We require that the chosen curves γ do not meet the nodal points. Thus, as before, the Weil–Petersson symplectic form extends to $\overline{\mathcal{M}}_{g,2n}^{\Gamma}$ due to the result of Wolpert [39].

Now we want to construct a map $f : \widehat{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,2n}^{\Gamma}$ in order to pull back the Weil–Petersson metric and then check that it satisfies the properties of the theorem. We will construct this map by gluing n pairs of pants with two cusps to the boundary components of X. Let $(X, p_1, \dots, p_n) \in \widehat{\mathcal{M}}_{g,n}$ with geodesic boundary components $\gamma_1, \dots, \gamma_n$, see Figure 5.4b. Now construct $Y \in \overline{\mathcal{M}}_{g,2n}$ by the follwoing procedure. Let Σ_i for $i = 1, \dots, n$ be a hyperbolic pair of pants with boundary lengths $(l_{\gamma_i}(X), 0, 0)$, i.e. a pair of pants with two cusps. In order to glue Σ_i to the boundary γ_i we need to give two points which are identified. On Σ_i we choose one of the cusps to correspond to β_{2i-1} such that we obtain a unique point on the boundary of Σ_i which lies on a geodesic perpendicular to the boundary and going up the cusp to β_{2i-1}. On γ_i we pick p_i. Identifying these two points we obtain a surface Y with genus g and $2n$ cusps. Thus we can define

$$\begin{aligned} f : \widehat{\mathcal{M}}_{g,n} &\longrightarrow \overline{\mathcal{M}}^{\Gamma}_{g,2n} \\ [X, p_1, \dots, p_n] &\longmapsto [Y, (\gamma_1, \dots, \gamma_n)]. \end{aligned}$$

This map is smooth since the hyperbolic structure is not modified but one adds hyperbolic pieces and because the choice of the gluing is unique and depends smoothly on the points p_i. The map f can be used to define a symplectic form $\widehat{\omega} = f^*\omega_{\mathrm{WP}}$ on $\widehat{\mathcal{M}}_{g,n}$.

By construction of the map, moving the points on the boundary corresponds to twisting the glued pair of pants around the attaching curve. Since it is part of a set of decomposing curves this corresponds to a Fenchel–Nielsen twist and we see that f is equivariant with respect to the two T^n-actions, i.e. on $\overline{\mathcal{M}}^{\Gamma}_{g,2n}$ the twist flow along the curves γ and on $\widehat{\mathcal{M}}_{g,n}$ the rotation of the marked points on the boundary. This means that if $t \in T^n$ acts on $\overline{\mathcal{M}}^{\Gamma}_{g,n}(L)$, then $f^*t = f^{-1} \circ t \circ f$ acts on $\widehat{\mathcal{M}}_{g,n}$ and we have $(f^*t)^*\widehat{\omega} = f^*t^*(f^{-1})^*f^*\omega = f^*t^*\omega = f^*\omega = \widehat{\omega}$ implying that $\widehat{\omega}$ is invariant under the T^n-action.

The same argument shows that the torus action has the moment map $f^*\frac{\mathcal{L}^2}{2} = \frac{l^2}{2}$. First the fundamental vector field of $e_i \in \mathbb{R}^n$ is (for $p \in \widehat{\mathcal{M}}_{g,n}$) given by

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} f^{-1} \circ \mathrm{tw}^{tl_{\gamma_i}}_{\gamma_i} \circ f\right)(p) = \mathrm{T}_{f(p)}f^{-1} \cdot \frac{\partial}{\partial\theta_i}(f(p)) = \left((f^{-1})_* \frac{\partial}{\partial\theta_i}\right)(p),$$

for $p \in X$ and where θ_i is the length-normalized Fenchel–Nielsen twist coordinate. Thus we need to check

$$\left(\mathrm{d}\left(f^*\frac{\mathcal{L}_\Gamma^2}{2}\right)\right)(p) = \left(-\mathrm{i}_{(f^{-1})_*\frac{\partial}{\partial\theta_i}} f^*\omega_{\mathrm{WP}}\right)(p), \tag{5.2}$$

where $\mathcal{L}_\Gamma$ is the moment map of the torus action on $\overline{\mathcal{M}}^{\Gamma}_{g,2n}(L)$ which we constructed in Section 4.1.2. For the left hand side of (5.2) we have $\left(f^*\mathrm{d}\frac{\mathcal{L}_\Gamma^2}{2}\right)(p)$ and for the right hand side

$$(\omega_{\mathrm{WP}})_{f(p)}\left(\mathrm{T}_pf \cdot \mathrm{T}_{f(p)}f^{-1} \cdot \frac{\partial}{\partial\theta_i}(f(p)), \mathrm{T}_pf\cdot\right) = \left(f^*(-\mathrm{i}_{\frac{\partial}{\partial\theta_i}}\omega_{\mathrm{WP}})\right)(p)$$

which are equal as $\frac{\mathcal{L}_\Gamma^2}{2}$ is the moment map for the T^n-action on $\overline{\mathcal{M}}^{\Gamma}_{g,2n}(L)$, see Lemma 4.16. Thus $f^*\frac{\mathcal{L}_\Gamma^2}{2}$ is the moment map for the T^n-action on $\widehat{\mathcal{M}}_{g,n}$. It is explicitly given by $\frac{1}{2}\left(l^2_{\beta_1}(X), \dots, l^2_{\beta_n}(X)\right)$ as the boundary curves β_i are mapped onto the curves γ_i. Thus the function $\frac{l^2}{2} = f^*\frac{\mathcal{L}_\Gamma^2}{2}$ is the moment map.

Now before we prove the symplectomorphism from Theorem 5.31 let us first understand that $\mathcal{L}_\Gamma^{-1}(L)/T^n \simeq \overline{\mathcal{M}}_{g,n}(L)$ is symplectomorphic. An element in $\mathcal{L}_\Gamma^{-1}(L)$ looks like in Figure 5.4b with the additional information that the curves γ_i have length L_i. The T^n-action rotates the pairs of pants with the two cusps around the curves γ_i. Thus when taking the quotient with respect to this action we obtain the moduli space of nodal curves with n boundary components. This is because the pair of pants has a single point as moduli space and the only freedom is the choice of the twisting. However, Dehn twists are identified as we are considering moduli spaces and the remaining twisting freedom corresponds to the T^n-action. Thus $\mathcal{L}_\Gamma^{-1}(L)/T^n$ is diffeomorphic to $\overline{\mathcal{M}}_{g,n}(L)$. Furthermore, the symplectic structure is defined in terms of the hyperbolic structure on the surface. Thus we get the same symplectic structure on both moduli spaces and we have that $\mathcal{L}_\Gamma^{-1}(L) \simeq \overline{\mathcal{M}}_{g,n}(L)$ are symplectomorphic. It is thus enough to show that $l^{-1}(L)/T^n \sim \mathcal{L}_\Gamma^{-1}(L)/T^n$. Since f is constructed such that it conserves the length of the curves γ_i we see that f restricts to a map $f : l^{-1}(L) \longrightarrow \mathcal{L}_\Gamma^{-1}(L)$. Now consider the following diagram.

$$\begin{tikzcd}
l^{-1}(L) \arrow[r, "f"] \arrow[d, "\pi"'] & \mathcal{L}_\Gamma^{-1}(L) \arrow[d, "\overline{\pi}"] & \\
l^{-1}(L)/T^n \arrow[r] & \mathcal{L}_\Gamma^{-1}(L)/T^n \arrow[r, "\simeq"] & \overline{\mathcal{M}}_{g,n}(L)
\end{tikzcd}$$

Since f is equivariant with respect to the T^n-actions it descends to a diffeomorphism on the quotient and the diagram commutes. Furthermore, call α the induced symplectic form on $l^{-1}(L)/T^n$ and β the one on $\mathcal{L}_\Gamma^{-1}(L)/T^n$. Then we have

$$\pi^*\alpha = \widehat{\omega}_{|_{l^{-1}(L)}} = (f^*\omega_{\mathrm{WP}})_{|_{l^{-1}(L)}} = f^*\left(\omega_{\mathrm{WP}}\right)_{|_{\mathcal{L}_\Gamma^{-1}(L)}} = f^*\overline{\pi}^*\beta,$$

which implies that $l^{-1}(L)/T^n \simeq \mathcal{L}_\Gamma^{-1}(L)/T^n \simeq \overline{\mathcal{M}}_{g,n}(L)$ are symplectomorphic. □

Now we see how the moduli spaces of Riemann surfaces with boundary appear as symplectic reductions of $\widehat{\mathcal{M}}_{g,n}$. Since the value of the moment map is connected to the length vector of the boundary component we see that via the Duistermaat–Heckman theorem we can write the volume of the reduced space in terms of other integrals. Let us first digress on the Duistermaat–Heckman theorem.

Theorem 5.32. *Let (M^{2n}, ω) be a symplectic orbifold with a Hamiltonian T^k-action and moment map $l : M \longrightarrow (\mathbb{R}^k)^*$. Then for two values ξ and $\xi_0 \in P_0 \subset P$, where P_0 is a free chamber of the moment polytope P, we have*

$$[\omega_\xi] = [\omega_{\xi_0}] + \langle \xi - \xi_0, c\rangle \in H^2(M_{\xi_0}, \mathbb{Q}), \tag{5.3}$$

where ω_ξ and ω_{ξ_0} denote the reduced symplectic forms at $M_\xi := l^{-1}(\xi)/T^k$ and for ξ_0 correspondingly. Here, c is the vector of Chern classes of the S^1-subbundles of the T^k-principle bundle $l^{-1}(\xi) \longrightarrow l^{-1}(\xi)/T^k$ for any ξ in the same free chamber.

Proof. See [14]. □

Corollary 5.33. *The symplectic volume of the reduced space (M_ξ, ω_ξ) is given by*

$$\int_{M_\xi} \frac{\omega_\xi^{n-k}}{(n-k)!} = \int_{M_{\xi_0}} \frac{(\omega_{\xi_0} + \langle \xi - \xi_0, c\rangle)^{n-k}}{(n-k)!}.$$

The idea is that the T^n bundles correspond to the products of the P_i-bundles which become in the limit the P_i'-bundles over the moduli space with boundary curves of length 0 which can then be related via the bundle isomorphism above to the S^1-subbundles $\mathcal{L}_i$.
Applying this to our case we obtain

Theorem 5.34. *For the volume of the moduli space we have*

$$V_{g,n}(L) = \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\le 3g-3+n}} C_\alpha^{g,n} L^{2\alpha},$$

where $L^{2\alpha} = L_1^{2\alpha_1} \cdots L_n^{2\alpha_n}$ *and*

$$C_\alpha^{g,n} = \frac{2^{m(g,n)|\alpha|}}{2^{|\alpha|}\alpha!(3g-3+n-|\alpha|)!} \int_{\overline{\mathcal{M}}_{g,n}} \psi^\alpha \omega_{WP}^{3g-3+n-|\alpha|},$$

where $m(g,n)$ *is* 1 *if* $(g,n) = (1,1)$ *or* 0 *otherwise and* $\psi^\alpha = \psi_1^{\alpha_1} \wedge \ldots \wedge \psi_n^{\alpha_n}$.

Proof. Consider all cases but $g = n = 1$. In order to use Corollary 5.33 let us first determine the objects we have. Since the T^n-action on $\widehat{\mathcal{M}}_{g,n}$ is free we can use $\xi_0 = \frac{1}{2}(\epsilon_1^2, \ldots, \epsilon_n^2)$ close to 0 and $\xi = \frac{1}{2}(L_1^2, \ldots, L_n^2)$. The Chern class of the T^n-bundle is $\sum_{i=1}^n c_1(P_i)e_i$. Thus $\langle \xi - \xi_0, c\rangle = \sum_{i=1}^n \frac{(L_i^2-\epsilon_i^2)}{2} c_1(P_i)$. Furthermore we have $\widehat{\omega}_L = \omega_{\mathrm{WP}}$ on $\overline{\mathcal{M}}_{g,n}(L)$ by Theorem 5.31 and the same for ϵ. Inserting the correct dimensions for our case we obtain

$$\begin{aligned}
V_{g,n}(L) &= \int_{\mathcal{M}_{g,n}(L)} \frac{\omega_{\mathrm{WP}}^{3g-3+n}}{(3g-3+n)!} \\
&= \int_{\overline{\mathcal{M}}_{g,n}(L)} \frac{\widehat{\omega}_L^{3g-3+n}}{(3g-3+n)!} \\
&= \int_{\overline{\mathcal{M}}_{g,n}(\epsilon)} \frac{\left(\widehat{\omega}_\epsilon + \sum_{i=1}^n \frac{(L_i^2-\epsilon_i^2)}{2} c_1(P_i)\right)^{3g-3+n}}{(3g-3+n)!} \\
&= \frac{1}{(3g-3+n)!} \sum_{j=0}^{3g-3+n} \binom{3g-3+n}{j} \int_{\overline{\mathcal{M}}_{g,n}(\epsilon)} \omega_{\mathrm{WP}}^j \left(\sum_{i=1}^n \frac{(L_i^2-\epsilon_i^2)}{2} c_1(P_i)\right)^{3g-3+n-j} \\
&= \frac{1}{(3g-3+n)!} \sum_{j=0}^{3g-3+n} \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=3g-3+n-j}} \binom{3g-3+n}{j} \frac{|\alpha|!}{2^{|\alpha|}\alpha!} \times \\
&\qquad\qquad \times \int_{\overline{\mathcal{M}}_{g,n}(\epsilon)} \omega_{\mathrm{WP}}^j (c_1(P_1))^{\alpha_1} \cdots (c_1(P_n))^{\alpha_n} (L^2-\epsilon^2)^\alpha \\
&= \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\le 3g-3+n}} \int_{\overline{\mathcal{M}}_{g,n}(\epsilon)} \omega_{\mathrm{WP}}^{3g-3+n-|\alpha|} (c_1(P_1))^{\alpha_1} \cdots (c_1(P_n))^{\alpha_n} \frac{(L^2-\epsilon^2)^\alpha}{2^{|\alpha|}\alpha!(3g-3+n-|\alpha|)!}.
\end{aligned}$$

In this expression, the bundles P_i and the Weil–Petersson symplectic form depend on ϵ. However, by taking the limit $\epsilon \longrightarrow 0$ we obtain the result, as noted in [25]. This is because we get in the limit

$$V_{g,n}(L) = \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\le 3g-3+n}} \int_{\overline{\mathcal{M}}_{g,n}} \omega_{\mathrm{WP}}^{3g-3+n-|\alpha|} (c_1(P_1'))^{\alpha_1} \cdots (c_1(P_n'))^{\alpha_n} \frac{L^{2\alpha}}{2^{|\alpha|}\alpha!(3g-3+n-|\alpha|)!}.$$

But since $c_1(P_i') = c_1(\mathcal{L}_i) = \psi_i$ (see Corollary 5.26) this is equal to

$$V_{g,n}(L) = \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\le 3g-3+n}} \int_{\overline{\mathcal{M}}_{g,n}} \omega_{\mathrm{WP}}^{3g-3+n-|\alpha|} \psi^\alpha \frac{L^{2\alpha}}{2^{|\alpha|}\alpha!(3g-3+n-|\alpha|)!}$$

For $g = n = 1$ the action of T^n on $\widehat{M}_{g,n}$ is not free because all tori with one boundary curve have an elliptic involution, i.e. they have a symmetry. Consider any closed non-peripheral simple geodesic and cut the surface along it. One obtains a pair of pants which consists of two isometric hyperbolic hexagons which can be interchanged and one obtains the same surface. Thus, when rotating the marked point on the boundary only by one half of the circumference one obtains the same element. However, if we take the same symplectic form this means that the torus spins twice as fast, i.e. the fundamental fields are twice those that they were without the symmetry. Thus we need to multiply the moment map by a factor of two which results in an additional overall factor of $2^{|\alpha|}$. $\square$

So we have computed the $C^{g,n}_{\alpha}$ from Proposition 4.39 in another way. Thus we can relate the definition of the $\langle \cdots \rangle$ from Section 4.3.4 to the intersection numbers. Recall that we defined

$$\langle \alpha_1, \ldots, \alpha_n \rangle_g = 2^{-m(g,n)} C^{g,n}_{\alpha} 2^{|\alpha|} \alpha!.$$

Thus, using Theorem 5.34 we see

$$\langle \alpha_1, \ldots, \alpha_n \rangle_g = \frac{2^{m(g,n)(|\alpha|-1)}}{(3g-3+n-|\alpha|)!} \int_{\overline{\mathcal{M}}_{g,n}} \psi^{\alpha} \omega_{\mathrm{WP}}^{3g-3+n-|\alpha|}.$$

Since the intersection numbers of the ψ classes in $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ are given by their integral over the Deligne–Mumford orbifold it is obvious that they correspond to the highest powers in the polynomial representation of $V_{g,n}(L)$. So, for $|\alpha| = 3g-3+n$ we get

$$\langle \alpha_1, \ldots, \alpha_n \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n}.$$

Therefore we conclude

Theorem 5.35. *All intersection numbers of the ψ classes in $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ can be recursively calculated via the recursion relation for the Weil–Petersson volumes of moduli spaces.*

Proof. Although we have already shown this we want to calculate it again and express the recursion relation in terms of intersection numbers. This will become useful in the next chapter. Using the calculations from the proof of Proposition 4.39 we see that the highest order terms (denoted by an additional $[\alpha]$) in $V_{g,n}(L)$ for $\alpha \in \mathbb{N}^n$ with $|\alpha| = 3g-3+n$ are

$$\mathcal{A}^{\mathrm{con}}_{g,n}(L)[\alpha] = \frac{1}{2} \sum_{\substack{i,j \in \mathbb{N} \\ i+j=\alpha_1-2}} \frac{\langle i, j, \alpha_2, \ldots, \alpha_n \rangle_{g-1} \alpha_1!}{2^{|\alpha|-\alpha_1+i+j} \alpha! i! j!} \frac{(2i+1)!(2j+1)!}{(2i+2j+4)!} \frac{L^{2\alpha}}{\alpha_1} \tag{5.4}$$

$$\mathcal{A}^{\mathrm{dcon}}_{g,n}(L)[\alpha] = \frac{1}{2} \sum_{I \subset \{2,\ldots,n\}} \sum_{\substack{i,j \in \mathbb{N} \\ i+j=\alpha_1-2}} \frac{(2i+1)!(2j+1)!}{(2i+2j+4)!} \frac{\langle i, \alpha_I \rangle_{g_1} \langle j, \alpha_{I^c} \rangle_{g_2} \alpha_1!}{2^{|\alpha|-\alpha_1+i+j} \alpha! i! j!} \frac{L^{2\alpha}}{\alpha_1} \tag{5.5}$$

$$\mathcal{B}_{g,n}(L)[\alpha] = \sum_{i=2}^{n} \binom{2\alpha_j + 2\alpha_1}{2\alpha_1} \frac{\langle \alpha_2, \ldots, \alpha_j + \alpha_1 - 1, \ldots, \alpha_n \rangle_g \alpha_1! \alpha_j!}{2^{|\alpha|-1} \alpha! (\alpha_1+\alpha_j)(\alpha_1+\alpha_j-1)!} \frac{L^{2\alpha}}{2\alpha_j + 2\alpha_1} \tag{5.6}$$

$$\frac{\partial}{\partial L_1} L_1 V_{g,n}(L)[\alpha] = \frac{(2\alpha_1+1)\langle \alpha_1, \ldots, \alpha_n \rangle_g}{2^{|\alpha|} \alpha!} \tag{5.7}$$

Simplifying this and putting it into the recursion relation (4.18) we obtain

$$\begin{aligned}(2\alpha_1+1)\langle \alpha_1, \ldots, \alpha_n \rangle_g = 2 & \sum_{\substack{i,j \in \mathbb{N} \\ i+j=\alpha_1-2}} \frac{\alpha_1!}{(2\alpha_1)!} \frac{(2i+1)!(2j+1)!}{i!j!} \langle i, j, \alpha_2, \ldots, \alpha_n \rangle_{g-1} \\ & +2 \sum_{I \subset \{2,\ldots,n\}} \sum_{\substack{i,j \in \mathbb{N} \\ i+j=\alpha_1-2}} \frac{\alpha_1!}{(2\alpha_1)!} \frac{(2i+1)!(2j+1)!}{i!j!} \langle i, \alpha_I \rangle_{g_1} \langle j, \alpha_{I^c} \rangle_{g_2} \\ & +2 \sum_{j=2}^{n} \frac{\alpha_1! \alpha_j! (2(\alpha_1+\alpha_j)-1)!}{(2\alpha_1)!(2\alpha_j)!(\alpha_1+\alpha_j-1)!}.\end{aligned}$$

Here we have used that for the second sum, i.e. the disconnected contribution, the genera of the terms are determined by the arguments via $i+|\alpha_I| = 3g_1 - 2 + n_1$ (and similarly for the second term) because we are only looking at top degree contributions in (5.5). Furthermore we need that for $(2n+1)!! := (2n+1)(2n-1)\cdots 3\cdot 1$ we have

$$(2n-1)!! = \frac{(2n)!}{2^n n!},$$

as can be easily checked. Thus we can write the recursion relation in terms of intersection numbers of ψ-classes as

$$\begin{aligned}(2\alpha_1+1)!!\langle \alpha_1,\dots,\alpha_n\rangle_g = \frac{1}{2}\sum_{I\subset\{2,\dots,n\}}\ \sum_{\substack{i,j\in\mathbb{N}\\ i+j=\alpha_1-2}} (2i+1)!!(2j+1)!!\langle i,\alpha_I\rangle_{g_1}\langle j,\alpha_{I^c}\rangle_{g_2}\\ +\frac{1}{2}\sum_{\substack{i,j\in\mathbb{N}\\ i+j=\alpha_1-2}} (2i+1)!!(2j+1)!!\langle i,j,\alpha_2,\dots,\alpha_n\rangle_{g-1}\\ +\sum_{j=2}^{n}\frac{(2(\alpha_1+\alpha_j)-1)!!}{(2\alpha_j-1)!!}\langle \alpha_2,\dots,\alpha_j+\alpha_1-1,\dots,\alpha_n\rangle_g.\end{aligned} \tag{5.8}$$

And we complete them with the two initial conditions $\langle 0,0,0\rangle_0 = 1$ and $\langle 0\rangle_1 = \frac{1}{24}$. □

5.1.4 The Witten conjecture

Using the fact that we have a recursion relation for the Weil–Petersson volumes of the moduli spaces and the connection between the coefficients of these volumes and the intersection numbers we can derive a recursion relation for those intersection numbers. We will introduce the generating function for the intersection numbers and show that this recursion relation is equivalent to a set of equations for this generating function. Afterwards we will explain very briefly how this is related to a solution of the Korteweg–de Vries differential equation.

Definition 5.36. The generating function of intersection numbers of genus g is defined as

$$F_g(t) := \sum_{\alpha=\{\alpha_i\}_{i\in\mathbb{N}}\subset\Phi} \langle\alpha\rangle_g \prod_{r\in\mathbb{N}, r\geq 0}\frac{t_r^{n_r}}{n_r!},$$

where t is a sequence $\{t_i\}_{i\geq 0}$ of real numbers, Φ is the space of unordered finite sequences of non-negative integers and arbitrary length, $n_r := |\{i\in\mathbb{N} \mid \alpha_i = r\}|$ and $\langle\alpha\rangle_g$ denotes the intersection number $\langle\alpha_1,\alpha_2,\dots\rangle_g$.

The total generating function is defined as the formal sum

$$F = \sum_{g=0}^{\infty}\lambda^{2g-2}F_g.$$

Remark 5.37. The variables $\{t_i\}_{i\in\mathbb{N}}$ keep track of the classes we intersect and λ keeps track of the genus.

Definition 5.38. Define the following differential operators on the space of functions of the variables t_i for $i = 0, \ldots$

$$\begin{aligned}
L_{-1} &= -\frac{\partial}{\partial t_0} + \frac{\lambda^{-2}}{2} t_0^2 + \sum_{i=1}^{\infty} t_{i+1} \frac{\partial}{\partial t_i} \\
L_0 &= -\frac{3}{2}\frac{\partial}{\partial t_1} + \sum_{i=1}^{\infty} \frac{2i+1}{2} \frac{\partial}{\partial t_i} + \frac{1}{16} \\
L_k &= -\frac{(2k+3)!!}{2^{k+1}} \frac{\partial}{\partial t_{k+1}} + \frac{1}{2^{k+1}} \sum_{i=0}^{\infty} \frac{(2i+2k+1)!!}{(2i-1)!!} t_i \frac{\partial}{\partial t_{i+k}} \\
&\qquad + \frac{\lambda^2}{2^{k+2}} \sum_{i=0}^{\infty} \sum_{d_1+d_2=k-1} (2d_1+1)!!(2d_2+1)!! \frac{\partial^2}{\partial t_{d_1} \partial t_{d_2}},
\end{aligned}$$

where $k \geq 1$.

Lemma 5.39. *The operators $\{L_k\}_{k \geq -1}$ span one branch of a representation of the Virasoro algebra, i.e. they satisfy*

$$[L_n, L_m] = (n-m) L_{n+m}$$

for all $n, m \geq -1$.

Proof. This can be shown by a direct but long computation, see [26]. □

Conjecture 5.40 (Witten)**.** *The exponential of the generating function of the ψ-intersections is annihilated by the differential operators L_k for $k \geq -1$, i.e.*

$$L_k \exp(F) = 0 \qquad \forall k \geq -1. \tag{5.9}$$

Remark 5.41. Edward Witten conjectured (5.9) in [37] in the year 1990. Maxim Kontsevich proved it in 1991 in [20] via a direct combinatorial calculation using matrix models. Since then there have been numerous different proofs of different kind of the Witten conjecture, see e.g [19] and [28].

Proof. The proof consists in calculating $L_k \exp(F)$ in each power of λ and t and then compare coefficients to see that it reduces to the recursion relation (5.8) for $\alpha_1 = k$. For $k \geq 1$ we first express the equation $L_k \exp(F)$ in terms of F

$$\begin{aligned}
L_k \exp(F) = \exp(F) \Bigg(&-\frac{(2k+3)!!}{2^{k+1}} \frac{\partial F}{\partial t_{k+1}} + \frac{1}{2^{k+1}} \sum_{i=0}^{\infty} \frac{(2i+2k+1)!!}{(2i-1)!!} t_i \frac{\partial}{\partial t_{i+k}} \\
&+ \frac{\lambda^2}{2^{k+2}} \sum_{i=0}^{n-2} (2i+1)!!(2n-2i-1)!! \left(\frac{\partial F}{\partial t_i} \frac{\partial F}{\partial t_{n-1-i}} + \frac{\partial^2 F}{\partial t_i \partial t_{n-1-i}} \right) \Bigg).
\end{aligned}$$

Now we compute the individual terms. Since we want to compare coefficients we shift the indices immediatly to get the coefficients in front of the same power. Let us explain a little bit more detailed what happens for the first term

$$\frac{\partial F}{\partial t_i} = \sum_{g \geq 0} \lambda^{2g-2} \sum_{\alpha \in \Phi} \langle \alpha \rangle_g \prod_{r \geq 0, r \neq i} \frac{t_r^{n_r}}{n_r!} \frac{t_i^{n_i - 1}}{(n_i - 1)!}.$$

We see that we need to shift $n_i \longmapsto n_i + 1$ which means that there must be one more entry $\alpha = i$ by definition of the n_i. Since we sum over all sequences nothing changes in the summation and we have

$$\frac{\partial F}{\partial t_i} = \sum_{g \geq 0} \lambda^{2g-2} \sum_{\alpha \in \Phi} \langle i, \alpha \rangle_g \prod_{r \geq 0} \frac{t_r^{n_r}}{n_r!}.$$

Similarly, if we multiply by some variable t_i we need to remove an $\alpha = i$. One obtains

$$t_i \frac{\partial F}{\partial t_j} = \sum_{g\geq 0} \lambda^{2g-2} \sum_{\alpha\in\Phi} n_i \langle \alpha_1, \dots, j, \dots, \alpha_n \rangle_g \prod_{r\geq 0} \frac{t_r^{n_r}}{n_r!},$$

where the n_i comes from the shift in the factorial. Note that this automatically takes care of the case $n_i = 0$, i.e. when there is nothing to be removed. Furthermore we have

$$\frac{\partial^2 F}{\partial t_i \partial t_j} = \sum_{g\geq 0} \lambda^{2g-2} \sum_{\alpha\in\Phi} \langle i, j, \alpha \rangle_g \prod_{r\geq 0} \frac{t_r^{n_r}}{n_r!}.$$

For the last term we need there is a small combinatorial issue. We have

$$\begin{aligned} \frac{\partial F}{\partial t_i} \frac{\partial F}{\partial T_j} &= \sum_{g_1, g_2 \geq 0} \lambda^{2(g_1+g_2)-4} \sum_{\alpha,\beta\in\Phi} \langle i, \alpha \rangle_{g_1} \langle j, \beta \rangle_{g_2} \prod_{r\geq 0} \frac{t_r^{n_r^\alpha + n_r^\beta}}{n_r^\alpha! n_r^\beta!} \\ &= \sum_{g\geq 0} \lambda^{2g-4} \sum_{\gamma\in\Phi} \sum_{\alpha\sqcup\beta=\gamma} \langle i, \alpha \rangle_{g_1} \langle j, \beta \rangle_{g_2} \prod_{r\geq 0} \frac{t_r^{n_r^\gamma}}{n_r^\gamma!} \frac{n_r^\gamma!}{n_r^\alpha! n_r^\beta!}, \end{aligned}$$

where g_1 is determined by the condition $i + |\alpha| = 3g_1 - 2 + n_1$ if $\alpha \in \mathbb{N}^{n_1}$ and correspondingly for g_2. In our case we have $j = k - 1 - i$ and thus we can rewrite this as

$$\sum_{g\geq 0} \lambda^{2g-4} \sum_{\alpha\in\Phi} \sum_{I\subset\{2,\dots,n\}} \langle i, \alpha_I \rangle_{g_1} \langle k-1-i, \alpha_{I^c} \rangle_{g_2} \prod \frac{t_r^{n_r}}{n_r!}.$$

Now putting everything together we obtain for $2^k L_{k-1} \exp(F)$ the expression

$$\begin{aligned} \exp(F) \sum_{g\geq 0} \lambda^{2g-2} \sum_{\alpha\in\Phi} \prod_{r\geq 0} \frac{t_r^{n_r}}{n_r!} \Bigg(&-(2k+1)!! \langle k, \alpha \rangle_g \\ &+ \sum_{i=0}^{\infty} \frac{(2i+2k-1)!!}{(2i-1)!!} n_i \langle \alpha_1, \dots, \alpha_{i+k-1}, \dots, \alpha_n \rangle_g \\ + \frac{1}{2} \sum_{i=0}^{k-2} (2i+1)!!(2k-2i-3)!! \Bigg(&\langle i, n-i-1, \alpha \rangle_{g-1} \\ &+ \sum_{I\subset\{2,\dots,k-1\}} \langle i, \alpha_I \rangle_{g_1} \langle k-2-i, \alpha_{I^c} \rangle_{g_2} \Bigg) \Bigg), \end{aligned}$$

where we have taken care of the genus shift in the last two terms. Due to the product of the two series we had $\lambda^{2g-4}\lambda^2$ which is the correct genus. Up to reordering, relabelling and $\alpha_1 = k-1$ this is exactly the recursion relation for the ψ-classes (5.8). Thus we have shown that $L_k \exp(F) = 0$ for $k \geq 1$.

It remains the case $k = -1$ and $k = 0$. It is enough to use the results from above to translate the equations $L_k \exp(F) = 0$ into equations for certain orders of F in λ and t. Note that the constant terms in L_1 and L_0 translate to constant terms for the differential equation for F. $L_{-1} \exp(F)$ is in all orders except $\lambda^{-2} t_0^2$ equal to

$$-\langle 0, \alpha \rangle_g + \sum_{i=1}^{\infty} \langle \alpha_1, \dots, i, \dots, \alpha_n \rangle_g,$$

where the last expression means that if one of the $\alpha_i = i+1$ then one replaces one of them by i.

This is equivalent to

$$\langle 0, \alpha \rangle_g = \sum_{\alpha_i \neq 0} \langle \alpha_1, \ldots, \alpha_i - 1, \ldots, \alpha_n \rangle_g$$

which is the dilaton equation (4.26). At order $\lambda^{-2} t_0^2$ we have

$$L_{-1} \exp(F) \left[\lambda^{-2} \frac{t_0^2}{2} \right] = \exp(F)(-\langle 0,0,0 \rangle_0 + 1)$$

because λ^{-2} translates into $g = 0$ and t_0^2 means that $\alpha = (0,0)$. However, this is just one of the initial conditions.
For $L_0 \exp(F)$ one obtains at any non-constant order

$$-\frac{3}{2} \langle 1, \alpha \rangle_g + \sum_{i=1}^{\infty} \frac{2i+1}{2} n_i \langle \alpha \rangle_g .$$

The last sum gives $\sum_{i=1}^{\infty} \left(i n_i + \frac{1}{2} n_i \right) = |\alpha| + \frac{1}{2} n = 3g - 3 + \frac{3}{2} n$ by definition of the n_i. Thus the equation is equivalent to

$$\langle 1, \alpha \rangle_g = (2g - 2 + n) \langle \alpha \rangle_g$$

which is nothing but the string equation (4.25). At constant order α is empty and the expression $L_0 \exp(F)$ reduces to

$$-\frac{3}{2} \langle 1 \rangle_1 + \frac{1}{16},$$

which is the second initial condition. Thus we have shown that the Witten conjecture is equivalent to the recursion relation which we had already proven □

Remark 5.42.

1. This statement is closely related to the fact that (5.9) is equivalent to $\exp(F)$ being a solution to the KdV hierarchy, which is an infinite set of differential equations coming from the theory of dynamical systems and which starts with the Korteweg-de Vries equation, i.e.

   $$\frac{\partial U}{\partial t_1} = U \frac{\partial U}{\partial t_0} + \frac{1}{12} \frac{\partial^3 U}{\partial t_0^3} .$$

   One formulation of the Witten conjecture is that $U = \frac{\partial^2 F}{\partial t_0^2}$ satisfies this equation as well as its generalization, see [28]. It can be shown that the KdV hierarchy as well as the string equation together with an initial condition $\langle 0,0,0 \rangle_0 = 1$ determine all intersection numbers and thus the function F. This we have already seen from the recursion relations. In this sense there is a correspondence between intersection theory on the moduli space, twodimensional quantum gravity and a dynamical system governed by the KdV hierarchy.

2. We have been specializing on the top degree contributions because we were interested only in the ψ intersections. However, the integrals we obtain for the lower degress, see (5.34), give intersections of the ψ classes with the class $[\frac{\omega_{\mathrm{WP}}}{2\pi^2}] \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. In fact, this class is another tautological class, usually denoted by κ_1, see [27] and [15]. It is possible to modify the differential operators such that we obtain the same result (i.e. the differential operators annihilate the generating functions and they obey a Virasoro algebra relation) for the generating functions of the ψ and κ_1 intersections. Unfortunately, the calculations become even more messy, see [26].

5.2 Topological string theory and quantum gravity

In this section we will describe roughly the setup of topological string theory and especially some objects which can be related to the ψ-classes we already know. This then in turns gives us a motivation for what the recursion relation and thus the determination of the intersection numbers of the ψ-classes can be used in physics. Let us first summarize the general picture of string theory. In string theory one considers maps from the circle S^1 to some arbitrary background space M evolving in time and therefore defined on some two-dimensional Riemannian manifold Σ. Thus it consists of fields (i.e. maps $\Sigma \longrightarrow M$) describing the position in the background (bosonic fields), some fermionic fields as the super partners of these as well as some ghosts which are necessary for the gauge fixing procedure. These fields are coupled to two-dimensional gravity which requires more fields and ghosts describing this theory as well as its gauge-fixing. Now we have several symmetries, especially supersymmetry on the world-sheet as well as conformal invariance. The last symmetry translates into the fact that the energy–momentum tensor is conserved. The action is chosen such that the equation of motion of the bosonic field is the wave equation. However, one chooses complex coordinates on the world-sheet such that it translates into $\partial\overline{\partial}X = 0$. The supersymmetric fields Ψ obey the two-dimensional Dirac equation, $\rho^\alpha\partial_\alpha\Psi = 0$. Solving these equations one then uses their Poisson brackets in order to determine their algebraic structure. Due to the choice of complex coordinates all the fields split into a holomorphic and an antiholomorphic part. The energy–momentum tensor can then be written as a sum over its holomorphic and antiholomorphic modes. These modes satisfy a Virasoro algebra. The fields are then promoted to operators-valued functions with respect to some Hilbert space. The energy–momentum tensor coming from local variations of the metric on the Riemann surface Σ is not invariant on the quantum level and thus causes an anomaly. This is solved by passing to a central extension of the Virasoro algebra and by restricting the Hilbert space to a certain subset and by taking some quotient. One is then able to compute the physical lowest-mass states and may try to find their transition probabilities. These values are defined by integrating some correlation function over the Riemann surface Σ after translating the states into some operator inserted at points on the surface. In order to make all the integrals finite one fixes a gauge slice and then introduces ghosts such that their integrals kill the contribution from the Jacobian coming from the change of coordinates due to the gauge-fixing procedure. These ghosts coming from certain gauge symmetries are then operator-valued functions themselves, satisfy a certain algebra and may appear in the construction of physical fields. Of course, it is pretty lengthy to formulate this in every detail, so we will restrict ourselves to the important ingredients in topological string theory, such that we can see how the intersection numbers arise in a physical context.

Before describing topological string theory in the next section we want to say a few words about its use. Since one is – after all – interested in some effective four-dimensional physical theory one usually considers the target space M to be a product of $\mathbb{R}^4$ and some compact six-dimensional manifold (e.g. a Calabi–Yau three-fold), where the number of 10 real dimensions comes from the anomaly described above. Thus, in order to make contact with four-dimensional physics one would like to investigate the effective theory on this $\mathbb{R}^4$. The world–sheet supersymmetry in fact results into a target-space supersymmetry which then restricts to a supersymmetric theory on the four-dimensional space. Since it is a $\mathcal{N} = 1$ SUSY in four dimensions it is effectively described by an action consisting of two functions, the Kähler- and the superpotential. Thus one is interested in determining these two potentials. All necessary data is encoded in the transition amplitudes. However, there is a theorem, see [32], that for the super potential it is in fact enough to know all transition amplitudes of the BPS-states of the supersymmetric theory as it is protected against variations of the coupling. However, these BPS-states can be determined by looking at a suitable BRST-cohomology, where the BPS-states correspond to some cohomology class. In fact one sees that their correlation functions are global quantities, i.e. they are not expectation values of operators inserted at certain points but just numbers, which is why one calls the theory describing this sector topological string theory.

5.2.1 Setup

We will now proceed and describe the basic ingredients we need in order to define a topological string theory. However, the description will be far from complete and many mathematical issues will be skipped. Furthermore we will leave some specifications open until some later time because many choices will become clear later. The main purpose is to give an outline why one may be interested in the cohomology of the moduli space and how the mathematical objects can be interpreted from a physical point of view. We will need the following three ingredients:

- matter
- Liouville gravity
- ghosts

Matter sector coupled to gravity

For the matter we are interested in a topological conformal field theory, it means a theory where we have a twisted $\mathcal{N} = 2$ supersymmetry ($\mathcal{N}$ denotes the number of independent supercharges) coming from conformal invariance and topological symmetries. One consequence is that correlation functions are independent of insertion points. We need a generator Q for the supersymmetry and a stress tensor T describing conservation of energy and momentum. These two fields can be used to define other fields which exhibit a twisted $\mathcal{N} = 2$ superconformal algebra. Bringing this into a standard form one twists the stress tensor T and obtains two supercurrents of conformal spin 1 and 2. Thus, if we have basic scalar fields x we will also have to include their super partners of spin 1 and 0. All these fields will be classically defined via geometric objects and then promoted to operators on some unfixed Hilbert space. The Q operator will then serve to determine the set of physical states, and, as it is the supersymmetry generator, the BPS states which are protected against changes of the couplings.
Thus one begins by considering a flat world-sheet, $\Sigma_0 := \mathbb{R}\times S^1$, and a target space M, supposed to be a complex manifold of a still to be determined dimension. One introduces the fields $x_i : \Sigma \longrightarrow \mathbb{C}$ for $i = 1, \dots, n$ with $2n = \dim M$ which are interpreted as coordinate functions of the string, i.e. they determine a point on M. Furthermore one wants to have two Q-supersymmetric partners of spin 1 and 0, χ_i^α and ψ_i. They are maps from Σ to a spin representation determined by their spin value, which is why the spin-1 field has an additional index.
Additionally we define a field which is a metric $g_{\alpha\beta}$ on M as well as its Q-partner $\psi_{\alpha\beta}$ which is a spin-2 gravitino. These two fields describe the gravitational sector. We now couple all fields via the following action functional

$$S[x_i, \psi_i, g_{\alpha\beta}, \psi_{\alpha\beta}] = \int_\Sigma \sqrt{g} g^{\alpha\beta}\eta^{ij}\left(\partial_\alpha x_i^* \partial_\beta x_j + \chi_{i,\alpha}\partial_\beta\psi_j + \widehat{\psi}_{\alpha\gamma}\chi_i^\gamma \partial_\beta x_j\right) \mathrm{d}\,\sigma \mathrm{d}\,\tau,$$

where $\widehat{\psi}_{\alpha\beta} := \psi_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\psi_\gamma^\gamma$ is the traceless part of the spin-2 gravitino, the coordinates τ and σ are the standard coordinates on Σ and the fields $g_{\alpha\beta}$ and $\psi_{\alpha\beta}$ depend on σ and τ through the x_i. This action exhibits various symmetries, namely diffeomorphism invariance (i.e. reparametrization of Σ, that is $\sigma, \tau \longmapsto \sigma', \tau'$), Weyl rescalings (i.e. $g \longmapsto \lambda g$) as well as their Q-superpartners and the Q-supersymmetry. However, we will not write down their infinitesimal versions but just refer to [8] for further details. One then defines scattering amplitudes following Polyakov by

$$\int_{\text{fields}} \mathcal{D}g_{\alpha\beta}\mathcal{D}\psi_{\alpha\beta}\mathcal{D}x_i\mathcal{D}\psi_i e^{-S[x_i,\psi_i,g_{\alpha\beta},\psi_{\alpha\beta}]} \prod_i \mathcal{O}_i,$$

where the integral is over the space of all fields (i.e. space of all sections or all maps with respect to some suitable measure) and the $\mathcal{O}_i$ are physical vertex operators, i.e. operators insterted at a point on the world-sheet corresponding to gauge-invariant states.

Gauge-fixing procedure and ghosts

Due to the different gauge symmetries we need to pick a gauge for the fields and integrate only over this slice because all other configurations are physically equivalent and give only infinite contributions. This gauge-fixing procedure includes a change of variables resulting in a Jacobian which needs to be cancelled by introducing new fields, so-called ghosts which cancel its contribution. Ignoring issues like global definability we can for example pick the superconformal gauge for the super-diffeomorphism invariance and a gauge for the super-Weyl transformations

$$
\begin{aligned}
g_{\alpha\beta} &= e^{\phi}\delta_{\alpha\beta} \\
\psi_{\alpha\beta} &= \psi e^{\phi}\delta_{\alpha\beta} \\
\overline{\partial}\partial\phi &= \widehat{R} \\
\overline{\partial}\partial\psi &= 0.
\end{aligned}
$$

Here, $\widehat{R}$ is some fixed curvature form and ϕ and ψ are the only remaining degrees of freedom. This results in the following gauge-fixed action

$$
\begin{aligned}
S &= S_{\text{matter}} + S_{\text{L}} + S_{\text{ghost}} \\
S_{\text{matter}} &= \int_{\Sigma} \eta^{ij}\left(\partial^{\alpha}x_i^{*}\partial_{\alpha}x_j + \chi_i^{\alpha}\partial_{\alpha}\psi_j\right) \mathrm{d}\sigma\, \mathrm{d}\tau \\
S_{\text{L}} &= \int \pi(\overline{\partial}\partial\phi - \widehat{R}) + \int \chi\overline{\partial}\partial\psi \\
S_{\text{ghost}} &= \int b\overline{\partial}c + \int \beta\overline{\partial}\gamma + \text{c.c.},
\end{aligned}
$$

where π and χ are spin-0 ghosts imposing the gauge-condition for the super-Weyl transformation and b and c are the anti-commuting spin-2 and spin-(-1) ghosts for the conformal gauge as well as β and γ their commuting Q-partners. Besides, c.c. stands for complex conjugate and refers to the fact that we neglect the right-moving sector. The last four ghosts are well known from usual string theory. We have now for each sector fields (including ghosts), an action and certain currents, i.e. expressions of the fields associated to the symmetry of the sector. All three sectors are topological conformal field theories and thus have a stress-tensor T, the symmetry Q and the two spin-1 and 2 currents J and G.

Although we did not write down the explicit infinitesimal symmetries we will assume the existence of the charges Q generating these symmetries and the corresponding currents $Q(z)$, where $Q = \oint Q(z)$. Notice that here z means that we write everything for the left sector only, bearing in mind that there is also a right sector, i.e. anti-holomorphic objects, denoted by an additional bar. For the currents of the matter sector we have

$$
\begin{aligned}
T_m(z) &= \{Q_m, G_m(z)\} \\
Q_m(z) &= -[Q_m, J_m(z)],
\end{aligned}
$$

where $T_m(z)$ and $Q_m = \oint Q_m(z)$ are defined by the action and $J_m(z)$ as well as $G_m(z)$ are defined via these equations. Its action is given by S_{matter} and its field multiplet is (x_i, ψ_i). In the sequel we will skip the dependence on the z-coordinate (i.e. the position on the world-sheet $z = \sigma + \mathrm{i}\tau$) of the currents of the individual sectors. It will be clear from the context and the equations wheter the object is the current or its charge, i.e. the integral of the current along the string.

The Liouville sector is given by the field content (ϕ, ψ, π, χ), its action is S_L and its symmety currents are

$$\begin{aligned}
T_L &= \partial\pi\partial\phi + \partial^2\pi + \partial\chi\partial\psi \\
Q_L &= \psi\partial\pi + \partial\psi \\
G_L &= \partial\chi\partial\phi + \partial^2\chi \\
J_L &= \psi\partial\chi + \partial\phi.
\end{aligned}$$

It remains the ghost sector. It is given by the fields (b, c, β, γ), the action S_{ghost} and its currents

$$\begin{aligned}
T_{\text{gh}} &= c\partial b + 2\partial cb + \gamma\partial\beta + 2\partial\gamma\beta \\
Q_{\text{gh}} &= b\gamma \\
G_{\text{gh}} &= c\partial\beta + 2\partial c\beta \\
J_{\text{gh}} &= bc + 2\beta\gamma.
\end{aligned}$$

5.2.2 Physical observables

The observables

Now we describe the BRST-like way to determine the BPS states of the theory as in [8].
We obtain first the total supersymmetry charge Q describing the supersymmetries which have to be implemented on the state space by integrating the currents above. This gives

$$Q_S = Q_m + \oint (b\gamma + \psi\partial\pi).$$

Furhtermore we have the superconformal symmetry and its Virasoro algebra. The Virasoro generators are given by the modes of the stress-tensor and the supercurrent G. One obtains

$$Q_V = \oint \left(c\left(T_m + T_L + \frac{1}{2}T_{\text{gh}} \right) + \gamma\left(G_m + G_L + \frac{1}{2}G_{\text{gh}} \right) \right).$$

One then chooses $Q_{\text{brst}} = Q_S + Q_V$ as the BRST-operator and defines the physical Hilbert space by

$$\mathcal{H}_{\text{phys}} := \frac{\ker Q_{\text{brst}}}{\text{im } Q_{\text{brst}}},$$

where Q_{brst} is restricted to the subspace of the Hilbert space $\mathcal{H}$ annihilated by the zero-modes of T and b of the right and left modes, i.e. $L_0 - \overline{L}_0$ and $b_0 - \overline{b}_0$. This ensures that the two sectors are coupled. Note that the term "physical" usually refers to the Q_V-cohomology as this kills all negative-norm states. However, here we are interested in the Q_S-cohomology as well because this gives us the protected BPS states. Thus we will refer to the Q_{brst}-cohomology as the physical state space.
Now it remains to look for states in $\mathcal{H}$ which are annihilated by Q_{brst} and which are not Q_{brst} exact. One such state is the so-called ghost vacuum given by

$$c_{-1}\overline{c}_{-1}\delta(\gamma_{-1})\delta(\overline{\gamma}_{-1})|0\rangle,$$

where $|0\rangle$ is a $\mathrm{SL}(2,\mathbb{C})$-invariant state. Using state–operator correspondence we associate to it a vertex operator, the so-called puncture operator

$$\mathcal{P} = c\overline{c}\delta(\gamma)\delta(\overline{\gamma}).$$

It can be shown that this state is indeed annihilated by Q_{brst}. Furthermore we define two operators

$$\gamma_0 := \frac{1}{2}(\partial\gamma + \gamma\partial\phi - c\partial\psi - \text{c.c.}),$$
$$c_0 := \frac{1}{2}(\partial c + c\partial\phi - \text{c.c.}),$$

which are not the 0-modes of γ or c. They are constructed out of ghosts solely, as is the puncture operator $\mathcal{P}$ and γ_0 can in fact be used to define a Q_{brst}-closed operator corresponding to a Q_{brst} invariant state, i.e. a physical state. Its associated operator is given by

$$\sigma_n := \gamma_0^n \mathcal{P}.$$

Since these operators are built from Liouville and ghost fields only, they correspond to pure gravitational physical operators and their correlation functions describe the underlying topological gravity theory. In order to relate these operators σ_n to the intersection numbers we have been investigating we now need to translate these operators via the superfield formalism to forms that we can integrate over the picked gauge slice.

Correspondence between fields, superfunctions and differential forms

Let us again switch to the more general case in which we also consider the matter sector. These arguments are based on [36] and [34]. As it turns out the equation of motion is in fact the non-linear Cauchy–Riemann equation which means that the solution space consists of J-holomorphic maps from some Riemann surface Σ to M. Thus the path-integral will achieve its biggest contribution from the integral over the moduli space of J-holomorphic maps form Σ to M. Let us call this moduli space $\mathcal{M}$. Since we also consider fermionic fields we have to integrate over fermionic field configurations as well, thus we have to extend the moduli space to the super moduli space $\widehat{\mathcal{M}}$ which includes these degrees of freedom, too. It can locally be described by bosonic coordinates a^λ and fermionic coordinates χ^λ, where λ runs from one to the dimension of the moduli space. The coordinates χ^λ (not to be confused with the above χ) transform as $\mathrm{d}a^\lambda$ as they are anticommuting variables. The idea behind this is that given solutions of the Cauchy–Riemann equation one can write any fermionic solution as a linear combination of the bosonic ones with anticommuting constant coefficients because any Taylor expansion of functions defined on anticommuting variables terminates as soon as the first variable appears squared.
Since a^λ and χ^λ transform oppositely (in the sense that $a^\lambda \longmapsto a^{\lambda'}$ induces $\frac{\partial}{\partial a^{\lambda'}} = \frac{\partial a^\mu}{\partial a^{\lambda'}}\frac{\partial}{\partial a^\mu}$ but $\mathrm{d}a^{\lambda'} = \frac{\partial a^{\lambda'}}{\partial a^\mu}\mathrm{d}a^\mu$) one can define a canonical measure on $\widehat{\mathcal{M}}$, namely

$$\mathrm{d}\mu = \prod_\lambda \mathrm{d}a^\lambda \prod_\lambda \mathrm{d}\chi^\lambda.$$

Now consider a function Φ on $\widehat{M}$, homogeneous of degree k in the anticommuting variables. It has the general form

$$\Phi = W_{\lambda_1 \dots \lambda:k}\chi^{\lambda_1}\cdots\chi^{\lambda_k}.$$

Since integration of such anticommuting functions with respect to $\mathrm{d}\mu$ is defined as the integral over the coefficient of top degree we see that

$$\int_{\widehat{\mathcal{M}}} \Phi \mathrm{d}\mu = \int_{\mathcal{M}} \widehat{\Phi},$$

where $\widehat{\Phi} = W_{\lambda_1\dots\lambda_k}\mathrm{d}a^{\lambda_1}\cdots\mathrm{d}a^{\lambda_k}$ is the corresponding form on the bosonic moduli space. In this way path integrals over field configurations of bosonic and fermionic fields correspond to integrals of certain forms over the moduli space.

General correspondence

Now we want to investigate two more principles. First, how the BRST-quantization gives us physical operators which can be interpreted as forms on the target space M and furthermore how these forms can be transformed into forms on the moduli space. For the moment call $Q := Q_{\mathrm{brst}}$.

Suppose we are given a n-form $A = A_{i_1 \dots i_n} \mathrm{d}u^{i_1} \cdots \mathrm{d}u^{i_n}$ on the target space M. Then consider an operator $\mathcal{O}_A = A_{i_1 \dots i_n} \chi^{i_1} \cdots \chi^{i_n}$. Now it is possible to show that such a state transforms under BRST-symmetries, which are generated by $\{Q, \cdot\}$, as $\delta \mathcal{O}_A = \mathrm{i}\epsilon \partial_{i_0} A_{i_1 \dots i_n} \chi^{i_1} \cdots \chi^{i_n}$ implying that

$$\{Q, \mathcal{O}_A\} = -\mathcal{O}_{\mathrm{d}A},$$

which means that $H_{\mathrm{brst}} \simeq H_{\mathrm{dR}}(M)$.

One can argue that the correlation function $\langle \mathcal{O}_{A_1}(P_1) \cdots \mathcal{O}_{A_n}(P_n) \rangle$, where $P_1, \dots, P_n$ are disjoint points on Σ are invariant under infinitesimal changes of the metric on Σ and are thus also independent of the choice of the points P_i. Thus, the correlation functions are in fact constants. This is closely related to the fact these functions are invariant under perturbations of the metric which is again related to the definition of topological field theories. However, this means that the difference of an operators inserted at two distinct points is Q_{brst}-exact and after some calculation

$$\mathrm{d}\mathcal{O}_A = \mathrm{i}\{Q, \mathcal{O}'_A\},$$

where $\mathcal{O}_A$ is seen as an operator-valued zero-form on Σ. Then

$$\mathcal{O}_A(P) - \mathcal{O}_A(P') = \{Q, \int_\gamma \mathcal{O}'_A\}$$

for a path γ from P' to P. This suggests that $W_A(\gamma) := \int_\gamma \mathcal{O}'_A$ with γ a homology cycle gives new observables for each closed one-form A and each homology cycle γ.

Now let us generalize this to see how to get a correspondence between general forms on the moduli space and physical states. Consider the universal instanton $\alpha : \Sigma \times \mathcal{M} \longrightarrow M$, where $\mathcal{M}$ is again the moduli space of J-holomorphic maps $\Sigma \longrightarrow M$, seen as a parameter space for those maps. Then α is the evaluation, i.e. $(z, \phi) \longmapsto \phi(z)$. For each closed differential form A of arbitrary degree and some homology cycle on M one defines $\widehat{\Phi}(A, \gamma) = \int_\gamma (\alpha^* A)_{|\gamma \times \mathcal{M}}$. Then one has

$$\begin{aligned} \langle \prod_{i=1}^n W_{A_i}(\gamma_i) \rangle &= \int_{\mathcal{M}} \widehat{\Phi}(A_1, \gamma_1) \wedge \ldots \wedge \widehat{\Phi}(A_n, \gamma_n) \\ &= \int_{\widehat{\mathcal{M}}} \Phi(A_1, \gamma_1) \cdots \Phi(A_n, \gamma_n), \end{aligned}$$

where $\Phi(A, \gamma)$ is the function on $\widehat{\mathcal{M}}$ corresponding to $\widehat{\Phi}(A, \gamma)$ on $\mathcal{M}$. As this is possible for all degrees of forms and homology cycles this generalizes the procedure above. Of course one still needs to write this in a strict way and one needs to generalize the argument why these forms actually correspond to physical observables, i.e. why they are Q_{brst}-invariant.

Summarizing we see that calculating expectation values of physical purely gravitational states corresponds to integrating products of certain cohomology classes over the moduli space. But this is exactly the cohomological definition of intersection products. It remains to identify the exact cohomology classes representing the gravitational states σ_n.

5.2.3 Relation to Chern classes and the Witten conjecture

Correspondence with Chern classes

In this chapter we will argue why the expectation value of the purely gravitational operators σ_n can be related to the intersection product of the ψ-classes, for more details see [8].
Recall that we have the operators σ_n, given by the ghost-operators γ_0 and that we want to relate their expectation values to the integral of certain Chern classes over moduli space. At the moment we know how to translate between geometric and physical picture in a general fashion but explicit calculations are hard.
Remembering that we defined the operators γ_0 and c_0 in terms of the ghosts we can calculate the explicit action of Q_S and Q_V on some operators. For example one obtains

$$\gamma_0 = \{Q_S, c_0\} \tag{5.10}$$

$$c_0 = \left[\frac{1}{2}(Q_V - \overline{Q}_V), \phi\right] \tag{5.11}$$

Now we have that $Q_{\text{brst}} = Q_V + Q_S$ and we saw above that in the geometrical picture Q_{brst} corresponds to the exterior derivative on M. Pulling back everything to forms on the moduli space we will see below that Q_S corresponds to the exterior differential d on the moduli space and Q_V to the Dolbeault operator ∂ on the moduli space. If we accept this we can see that (5.10) and (5.11) have a very special meaning. Remember that we choose a gauge slice by considering the metric $e^{\phi}\mathrm{d}z\mathrm{d}\overline{z}$ on Σ with a special condition on ϕ. Since the tangent bundle to Σ is a complex line bundle this metric induces some connection, namely the following

$$\Gamma = \frac{1}{2}(\partial - \overline{\partial})\phi. \tag{5.12}$$

By pulling back everything to the moduli space we also pull back the tangent bundle and obtain a complex line bundle over the moduli space. The reason that the puncture operator (used to create the states σ_n) is called in this way is that it generates a puncture in the sense of the geometrical picture. Comparing (5.12) and (5.11) one can see that the operator c_0 at a point $z_i \in \Sigma$ corresponds to the connection form of the pull-backed complex line bundle over the moduli space. But then (5.10) means that γ_0 corresponds to the curvature form of this connection. And thus we see that its physical state (in the BRST-cohomology) corresponds to the de-Rham cohomology class of the curvature form on the moduli space which is the Chern class of the complex line bundle which we called ψ.
So we can see that the calculation of these physical correlation functions of purely gravitational states can be related to the calculation of intersection numbers of ψ-classes on the moduli space which we have solved in the foregoing chapters. Under the assumption that these states are all purely gravitational states we conclude that the recursion relation (5.8) determines all expectation values between such states.
So it remains to understand the correspondence between Q_S and d and Q_V and ∂. One can show this for example by showing that the correlation functions in both pictures satisfy the same relations, see [8], i.e.

$$\langle \sigma_{n_1} \prod_{i=2}^{s} \sigma_{n_i} \rangle_\Sigma = \partial\overline{\partial}\phi(z_1) \wedge \langle \sigma_{n_1 - 1} \prod_{i=2}^{s} \sigma_{n_i} \rangle_\Sigma, \tag{5.13}$$

This correspondence can be shown by investigating the path-integral

$$\langle \sigma_{n_1} \cdots \sigma_{n_s} \rangle_\Sigma = \int \mathrm{d}\pi\, \mathrm{d}\phi\, \mathrm{d}\chi\, \mathrm{d}\psi\, \mathrm{d}b\, \mathrm{d}c\, \mathrm{d}\beta\, \mathrm{d}\gamma\, e^{-S} \gamma_0^{n_1} \mathcal{P}(z_1) \cdots \gamma_o^{n_s} \mathcal{P}(z_s).$$

The pull-back argument can be used to see that the exterior derivative on M corresponds to the exterior derivative on the moduli space and thus $Q_S \sim \mathrm{d}$. Then one shows

$$\langle \sigma_{n_1} \prod_{i=2}^{s} \sigma_{n_i} \rangle_\Sigma = \langle \{Q_S, c_0 \sigma_{n_1-1}\} \prod_{i=2}^{s} \sigma_{n_i} \rangle_\Sigma$$

and decomposing the ghost c_0 in a suitable way then

$$\langle c_0 \sigma_{n_1-1} \prod_{i=2}^{s} \sigma_{n_i} \rangle_\Sigma = \frac{1}{2}(\partial - \overline{\partial})\phi(z_1) \wedge \langle \sigma_{n_1-1} \prod_{i=2}^{s} \sigma_{n_i} \rangle_\Sigma.$$

Putting everything together one arrives at (5.13), see [8].

Witten conjecture

Recall that we also investigated the generating function $F(t_0, \ldots)$ of ψ-intersections for all genera. In the physical context the parameters t_i are interpreted as coupling constants of infinitely many possible string theories in which one changes the action by adding certain integrals proportional to the parameters t_i, i.e.

$$S \longmapsto S - \sum_n t_n \int \sigma_n$$

The function F is then called string free energy. Its exponential is the full string partition function and satisfies some Virasoro equations $L_n F = 0$ for all $n \geq -1$. This is when one gets in contact with matrix models describing quantum gravity, as one has the same equations for the free energy there, see [9].

6 Conclusions

In the last four chapters we have seen a very striking interplay between several fields of mathematics and physics. We began with hyperbolic geometry which we used to prove an identity for the lengths of closed simple curves in dependence on the length of the boundary geodesics of the surface. We then used it and integrated it over the moduli space of Riemann surfaces and were able to derive from this a recursion relation enabling us to calculate the Weil–Petersson volumes of moduli space. In the last chapter we showed that we could relate the volumes to intersection numbers of Chern classes of tautological bundles over the moduli space, the so-called ψ-classes. By pointing towards a correspondence between mathematics and physics we were thus able to calculate all expectation values of certain operators in topological string theory corresponding to topological gravity.
As was pointed out in the introduction there remained a few questions, namely the exact formulation in terms of orbifolds, the explicit construction of the tautological bundles over moduli space and the precise meaning of the limit used in the Duistermaat–Heckmann calculation.
We now want to take a look at a natural generalization of the Witten conjecture which was already mentioned in the introduction, the so-called Virasoro conjecture. Consider the compactified moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of J-holomorphic curves from a Riemann surface $\Sigma_{g,n}$ to a projective variety X representing some homology class $\beta \in H_2(X,\mathbb{Z})$. Furthermore let $\{\gamma_a\}$ with $\gamma_a \in H^{p_a,q_a}(X,\mathbb{C})$ be a basis of $H^*(X,\mathbb{C})$. Then the descendent Gromov–Witten invariants of X are defined as

$$\langle \tau_{k_1}(\gamma_{a_1})\cdots\tau_{k_n}(\gamma_{a_n})\rangle^X_{g,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \psi_1^{k_1}\mathrm{ev}_1^*(\gamma_{a_1})\cdots\psi_n^{k_n}\mathrm{ev}_n^*(\gamma_{a_n}),$$

where the ψ_i are the tautological classes in Gromov–Witten theory and ev denotes the evaluation map from the moduli space to X. Here we can see how this restricts to the case considered in this thesis for $X = \mathrm{pt}$. Then $H^*(X,\mathbb{C}) = 0$ and the γ_a all vanish and $\overline{\mathcal{M}}_{g,n}(X,\beta) = \overline{\mathcal{M}}_{g,n}$. However, the corresponding generating function for these variables can be written down as before, let us denote it by $F^X(t,\lambda)$. It is given by

$$F^X = \sum_{g\geq 0}\lambda^{2g-2}\sum_{\beta\in H_2(X,\mathbb{Z})} q^\beta \sum_{n\geq 0}\frac{1}{n!}\sum_{\substack{a_1\ldots a_n\\ k_1\ldots k_n}} t^{a_1}_{k_1}\cdots t^{a_n}_{k_n}\langle\tau_{k_1}(\gamma_{a_1})\cdots\tau_{k_n}(\gamma_{a_n})\rangle^X_{g,\beta}.$$

One then defines the operators L_k for $k \geq -1$ by

$$\begin{aligned} L_k = &\sum_{m=0}^{\infty}\sum_{i=0}^{k+1}\left([b_a+m]^k_i (C^i)^b_a \tilde{t}^a_m \partial_{b,m+k-i} + \frac{\hbar}{2}(-1)^{m+1}[b_a-m-1]^k_i (C^i)^{ab}\partial_{a,m}\partial_{b,k-m-i-1}\right) \\ &+ \frac{\lambda^{-2}}{2}(C^{k+1})_{ab}t^a_0 t^b_0 + \frac{\delta_{k,0}}{48}\int_X \left((3-r)c_r(X) - 2c_1(X)c_{r-1}(X)\right), \end{aligned}$$

where for $\gamma_a \in H^{p_a,q_a}(X,\mathbb{C})$ one has $b_a = p_a + \frac{1-r}{2}$, $[x]^k_i := e_{k+1-i}(x, x+1, \ldots, x+k)$ with e_k the k-th elementary symmetric function, C the matric defined by the cup product with the anticanonical class $c_1(X)$, i.e. $(C)^b_a\gamma_b = c_1(X)\cup\gamma_a$, the indiced of C are lowered and raised with the intersection pairing $g_{ab} = \int_X \gamma_a\cup\gamma_b$ and $\tilde{t}^a_m := t^a_m - \delta_{a,0}\delta_{m,1}$ and $\partial_{a,m} := \frac{\partial}{\partial t^a_m}$. This is, as one would expect, more complicated than in the case of the Witten conjecture. Now the Virasoro conjecture states that $L_k \exp(F^X) = 0$ for all non-singular projective varieties X. This conjecture is, in its full generality, unknown. However, it was already proven by Okounkov and Randharipande for complex curves and by Givental for $\mathbb{C}P^n$. In fact, in [28] they prove the Virasoro conjecture for the sphere and recover a correspondence similiar to the one mentioned

in Remark 5.42. For the sphere, the generating function for the intersections of the tautological classes is a solution to the Toda hierarchy from the theory of dynamical systems. In this sense one may believe that there is some unknown correspondence between Gromov–Witten theory and dynamical systems.
Differences to the case of $X = \mathrm{pt}$ are e.g. that the moduli space is much more complicated as it may not even be an orbifold. The definition of the ψ-classes is again causing problems and the operators depend on various objects defined on X. A direct try of similar concepts as we have used for the Witten conjecture is difficult as there is no Teichmüller space of J-holomorphic curves, there are no Fenchel–Nielsen coordinates, there is no Weil–Petersson metric and as we would have to consider surfaces with boundary we needed boundary conditions for the J-holomorphic curves. The last point leads to the consideration of open Gromov–Witten theory which depends on certain choices for boundary conditions which may again cause problems. We see that the more general conjecture is much more complicated and must involve many new concepts which we have not developed here. This can be considered to be one of the big problems in symplectic geometry and algebraic geometry, see e.g. [29].

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Statement of authorship

I hereby certify that this master thesis has been composed by myself, and describes my own work, unless otherwise acknowledged in the text. All references and verbatim extracts have been quoted, and all sources of information have been specifically acknowledged. It has not been accepted in any previous application for a degree.