MTH330: Bonus Homework

Due: Thursday 06/28/12

Problem 1 *Golden section*

The golden section is a real number which appears pretty often in various different contexts. It was first discovered by the ancient greeks and entered arts, architecture, mathematics, even biology and finance (well, supposingly). There are quite a lot of urban (and non-urban) legends about it, e.g. that the solar plexus of the human body is at the golden ratio of the human body height. However, it is indeed fascinating and is often cited as an example of the "application" of mathematics to the real world. In mathematics it can, for example, be used to give an explicit formula for the Fibonacci series (i.e. $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$) and appears as one of the "easy" continued fractions (i.e. the one given by $[1; 1, 1, 1, 1, 1, \ldots]$). Here we will mostly talk about its geometric definition, some properties and the associated polynomial. For more (non-mathematical) background you can have a look at the Wikipedia page on the golden section.

(a) Consider a line segment AB. We say that a point P on AB is at the golden section if the ratio of AP to PB is the same as the ratio of AB and AP. Calculate AP if $AB = 1$.

(b) Consider a rectangle $ABCD$ whose sides have this ratio. Consider a point P on the long side of the rectangle at at the golden section of AB. Draw PQ parallel to AD meeting DC at Q. Prove that $BCQP$ is similar to $ABCD$.

(c) Show that the drawing of PQ divided the rectangle into a rectangle and a square.

(d) If AC and BQ meet at O, prove that AC and BQ are perpendicular and that $\frac{AO}{OB} = \frac{BO}{OC} = \frac{CO}{OQ}$.

(e) Show that in a regular pentagon $ABCDE$, if any diagonal AD is met by two other diagonals EB and EC at P and Q respectively, P divides AQ in golden section, and Q divides DP in golden section.

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Problem 2 *Excircles*

This problem is about the construction of the excircles of a triangle. Recall that the excircles are defined as the circles touching (i.e. being tangent to) two extended sides and one side of the triangle and are outside of the triangle. Thus there are three different circles, see Fig. 2.

(a) Suppose we had constructed one of the excircles. Draw one of the lines from a vertex of the triangle to the center of one of the corresponding excircles. Show that it is the angle bisector of the corresponding exterior angle.

(b) Notice that this statement holds for all pairs of corresponding vertices and excircle-centers. Knowing this, how can you construct the excircle-center (also called an excenter) of a triangle?

(c) Construct a triangle with sides of lengths $1\frac{1}{2}$in, 2in and $2\frac{1}{2}$in. Construct all three excircles.

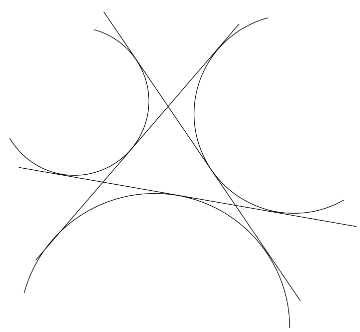

Figure 1:

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Problem 3 *Cheat sheet*

This is a list of figures which are related to the main lemmas and theorems which you need in elementary geometry to solve problems. Try to find the statements of all the figures in a short, memorizable and precise way. If possible try to write down the statement as well as its converse. If the theorem has a name try to give it.

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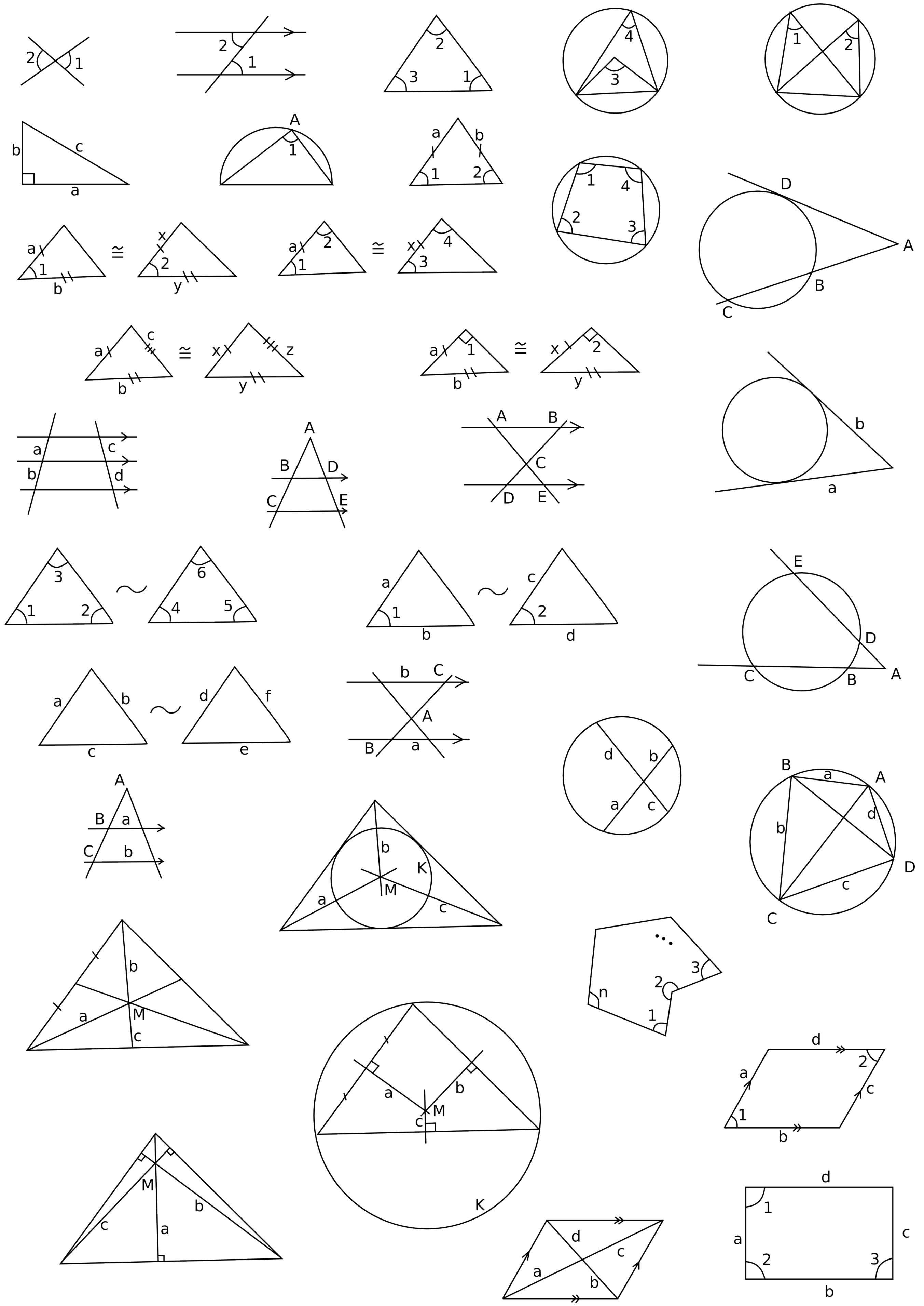

2 1
2 1
2 3 1
4 3
1 2
b c a
A 1
a b 1 2
1 4 2 3
D A B C
a 1 b ≅ x 2 y
a 2 1 ≅ x 4 3
c a b ≅ x z y
a 1 b ≅ x 2 y
a c b d
A B D C E
A B C D E
b a
3 1 2 ∼ 6 4 5
a 1 b ∼ c 2 d
E D C B A
a b c ∼ d f e
b C A B a
d b a c
B a A d b D c C
A B a C b
b K a M c
3 2 n 1
b a M c
a M b c K
d a 2 c 1 b
M c b a
d c a b
d 1 a c 2 3 b

Problem 4 *Morley's theorem*

Let us trisect the interior angles in a triangle and consider the three intersection points P_1, P_2 and P_3 of the angle-trisectors nearest to the sides of the triangle. We will prove that the triangle formed by these three points is always (i.e. for any given triangle that we start with) equilateral.

There are many different proofs, most of them fall in two categories: they are either trigonometric or use some kind of backwards argument ("cheating"). We sort of have seen this in the lecture whenever we proved statements by doing a construction with the property we wanted to show and then proving that the constructed object is indeed the one we actually talked about. See for example the proofs of concurrence of triples of lines in the triangle. Here we will do something similar. The proof goes as follows:

Suppose we are given a triangle with interior angles 3α, 3β and 3γ. Then we construct an arbitray equilateral triangle and use the angles α, β and γ to construct isosceles triangles outside of the equilateral one. Then we show that the resulting big triangle is similar to the given one which means that the equilateral triangle is similar to the Morley triangle of the given triangle. But this implies that the Morley triangle is equilateral. Here is the step by step instruction:

Suppose we are given a triangle $\triangle ABC$ with interior angles $3\alpha, 3\beta$ and 3γ. Construct an arbitray equilateral triangle.

(a) Prove that $\alpha + \beta + \gamma = 60\,°$.

(b) Construct isosceles trapezoids on the sides of the equilateral triangle with base angles 2α, 2β and 2γ and with three sides equal to the side of the equilateral triangle, see Fig. **??**. Denote their endpoints as in the figure. Use the neighboring pairs of the endpoints of the trapezoids to construct three lines which intersect in three points A', B', C'. Show that the angle $\angle C_0B_1B_0 = \alpha$. *Hint: You can show this in general: The diagonal of an isosceles trapezoid where three sides are equal bisects the base angles.*

(c) Calculate the angles $\angle C_0B_0B_1$, $\angle A_0B_0B_2$, $\angle B_2B_0B_1$, $\angle B_2B_1B_0$ and $\angle AB_1B_0$. Find $\angle C_0C_2A$ by arguing that this is the same calculation with some points interchanged. *Hint: Although this looks messy it is nothing but an unknown-angle problem. Just in reality this time.*

(d) Calculate $\angle C_0C_2A'$ and $\angle C_0B_1A'$. Conclude that the quadrilateral $A'C_2C_0B_1$ is cyclic.

(e) Argue that the same holds for the quadrilateral $A'C_0B_0B_1$. Show that this implies that the pentagon $A'C_2C_0B_0B_1$ is cyclic.

(f) Prove that $A'B_0$ and $A'C_0$ trisect the angle $\angle B_1A'C_2$. Calculate this angle.

(g) Note that this argument can be applied to all three vertices. What does this mean for the interior angles of the triangle $\triangle A'B'C'$? What is its relation to the original triangle $\triangle ABC$?

(h) Give a reason why this finishes the proof or show how the argument continues in order to prove the statement.

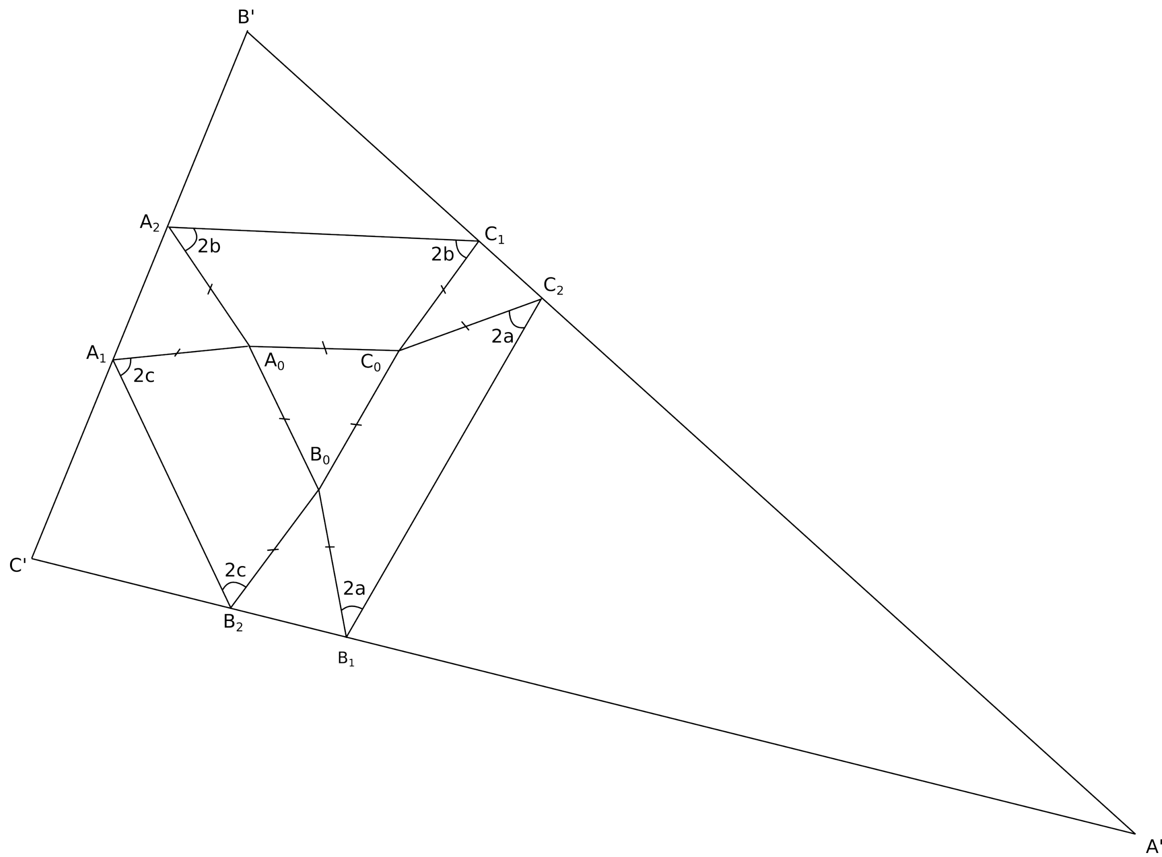

Figure 2:

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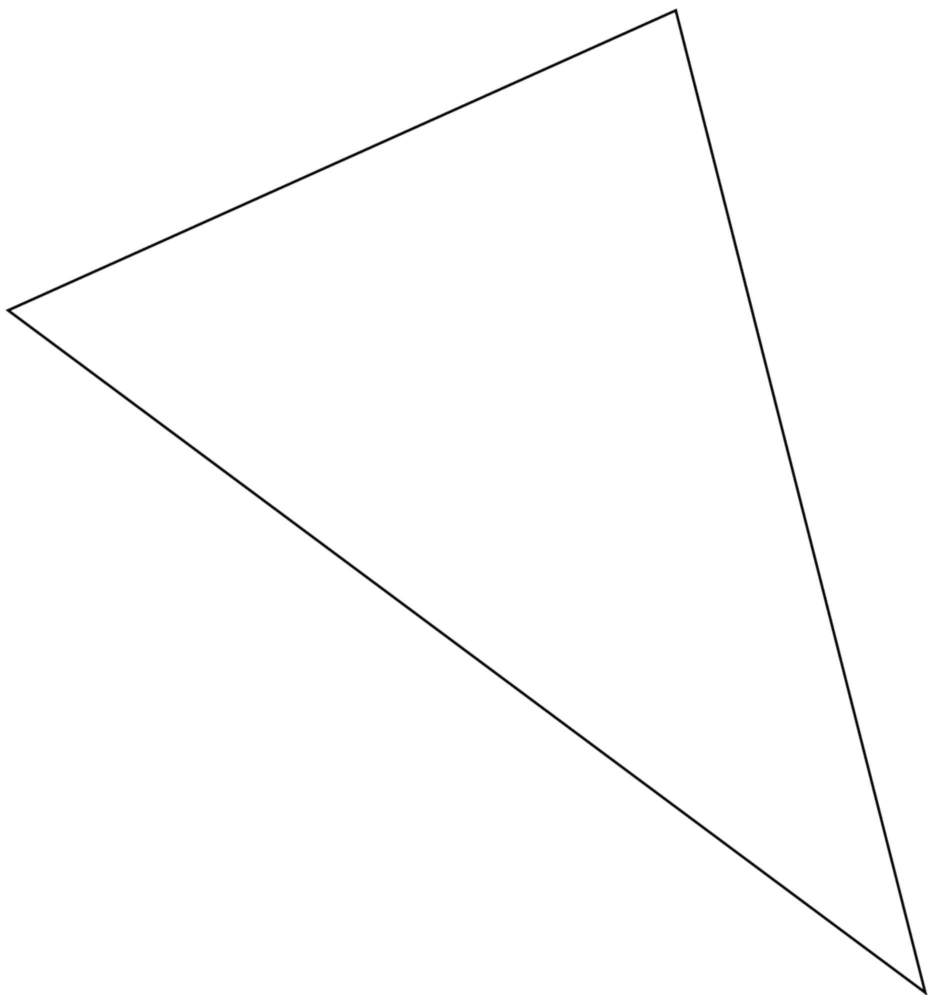
Problem 5 *Construction of Euler line and Feuerbach circle*

In the following triangle construct the Euler line and the Feuerbach circle. Recal their definitions from the lecture and remember to only use compass and ruler. Take care and be very precise!



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Problem 6 *Nagel point*

In the lecture we encountered four triples of lines through the vertices of a triangle which are concurrent. This problem is about another such triple which is far less known. The intersection point is called the Nagel point. Recall that we also talked about the theorem of Ceva which is a sufficient and necessary condition for three lines through vertices of a triangle to be concurrent (up to a small ambiguity). In the proof of the existence of the Nagel point we will use Ceva.

Consider a triangle $\triangle ABC$, where we call the sides a, b, c as usual. Then we have the three excircles meeting each two extended sides and one side of the triangle. Call the contact points as in Fig. 6. We will do some small calculations for one of the circles. The remaining identities will be deduced by arguing that the same proof shows them by renaming the important objects.

(a) Let's look at the lower part: What do you know about AF and AT? What about the pairs FB and BU as well as CT and CU? Think about something you have shown in another homework (and in the lecture).

(b) Show that $AC + AF = BC + BF$.

(c) Prove that $AF = s - b$ and $BF = s - a$, where s is the semi-perimeter.

(d) What are the corresponding equations for AE, CE, CD and BD? *Hint: Don't repeat the whole argument. Just look at the equalities and imagine what would happen if you "rotate" the argument.*

(e) Use Ceva's theorem to show that the lines AD, BE and CF are concurrent. This point is called the Nagel point.

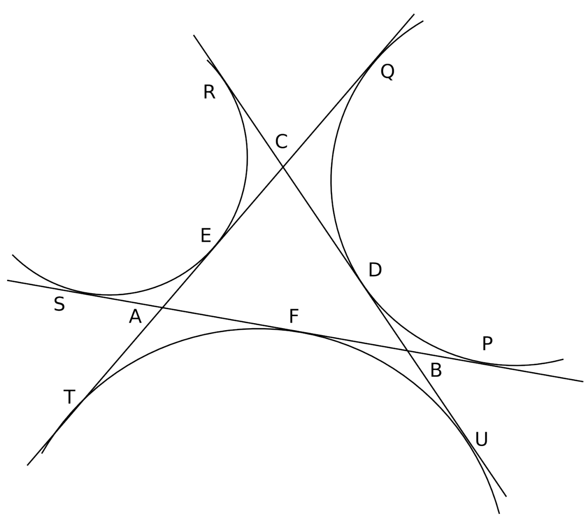

Figure 3:

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Problem 7 *Euler characteristic of spheres and tori*

In class we were talking about the property of a polyhedra that the number of vertices minus the number of edges plus the number of surfaces is equal to two. This number is called the Euler characteristic of the polyhedron. It is combinatorial, i.e. it doesn't depend on the length or the position of the individual objects and not on the particular form of these objects. For example, if we would bend the edges the number wouldn't change.

(a) So let us take any polyhedron and bend the edges as well as the surfaces such that the total polyhedron becomes smooth. Equivalently you can imagine to blow air into the interior of the polyhedron until you don't see the edges and vertices anymore. However they are still there, they just look differently. Which geometrical object do you obtain (forgetting the vertices and edges for a moment)?

(b) What is its Euler characteristic? *Hint: This question is essentially trivial.*

(c) Let us now not start with a polyhedron but with a sphere. If we want to calculate its Euler characteristic (or rather define it) we can try to find a (simple) polyhedron or some simple curved figure such that the operation from (a) gives the sphere. One easy example is in Fig. 7. Try to do the same with the torus (i.e. the surface of a donut, see Fig. 7).

(d) By counting the vertices, edges and surfaces, calculate the Euler characteristic of a torus.

(e) Repeat the same with a pretzel, see Fig. 7. What is its Euler characteristic? *Hint: This is not so easy to see. Try to start by putting a vertex into the "middle part" together with an edge such that the pretzel is divided into two parts. Then put another edge such that the two parts are cut into easier objects. Do this until all edges end at two (possibly the same) vertices and all surfaces look like a deformed disc with boundary some edges and vertices in between. When counting, remember to count each vertex seperatly (the figure should be simple, i.e. edges should only intersect in vertices) and all edges seperatly (one edge is the line segment between two vertices).*

(f) Can you guess a general formula for a pretzel with n holes? Here the torus has one hole and the pretzel two.

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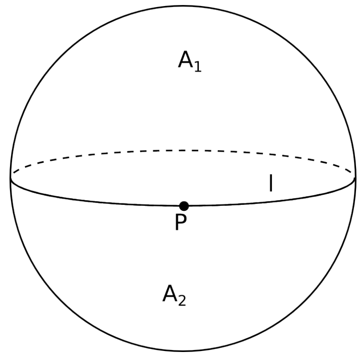

Figure 4:

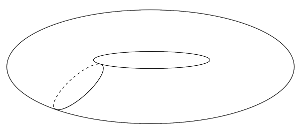

Figure 5:

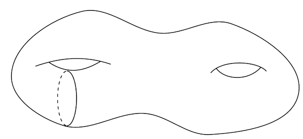

Figure 6:

Problem 8 *Geometry on the surface of the sphere*

In the Weeks & Adkins book this is studied in the last chapter beginning on page 472. In turns out in differential geometry that two-dimensional (finite) surfaces always admit some kind of "model geometry", whatever this means. There are essentially three: flat (Euclidean) space, hyperbolic space and spherical spaces. The first one is what we were investigating in detail in class, the second one is very important for many areas of mathematics and the spherical case is what we will be looking at now. In regard of problem (prob:euler), one can show that a sphere admits a spherical geometry (hence its name), the torus admits a flat structure (so it locally looks like the Euclidean plane) and the surfaces with more holes admit hyperbolic structures. There are many differences between the three geometries, some of which we will see here.

In spherical geometry the straight lines correspond to the great circles of the sphere. An angle is the figure formed by two rays (i.e. segments of great circles) beginning at a common point. Its numerical value is given as follows: Turn the sphere such that you look on the vertex from "above". Then the two great-circle segments look like lines to you. Therefore you can measure their angle as in the plane and this is defined to be the angle of two great circles, see Fig. 8.

In the following problems "prove" is not meant as strict as otherwise. It is enough to give convincing or obvious arguments (Well, they definitley have to convince me at least.).

(a) Why do two different great circles have exactly two points in common?

(b) In Euclidean geometry there are n-polygons for $n \geq 3$. Can you think of a new figure which exists on a sphere but not in the Euclidean plane?

(c) Pick a point of the sphere and call him "north pole". Take any great circle through this point and follow it until you reach the "equator" and mark this point. Then follow the equator some distance, mark a third point and then take the great circle back to the north pole. Then you have drawn a triangle on the sphere. What is its interior angle sum? Is there a difference to the Euclidean case?

(d) Consider the 2-gon you obtain by connecting the north pole and the south pole by two semi great circles. This figure is determined by its "opening angle" at the nort pole, see Fig. 8. Let us calculate angles in radian from now on and suppose the angle is α. Knowing that the total area of the sphere is 4π, what is the area of this 2-gon?

(e) It can be shown that the sum of interior angles in a spherical triangle is always bigger than π. If it is $\pi + E$, then $E > 0$ is called its excess. We want to show that the area of the spherical triangle is equal to E. To this end, extend the sides of the spherical triangle until you have three great circles. How many 2-gons do you get by this construction? Calling the interior angles of the triangle α, β and γ, what are the areas of the 2-gons?

(f) By adding all the areas and comparing the surface with the sphere and the triangle, deduce that the excess is equal to the area of a triangle.

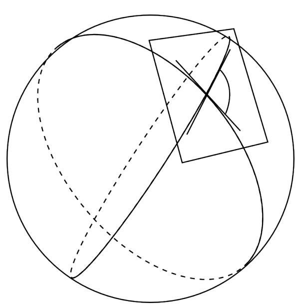

Figure 7:

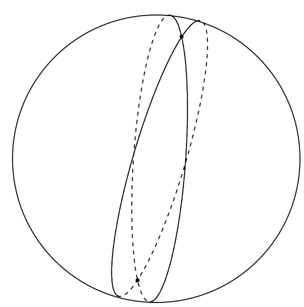

Figure 8:

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