MTH330: Problem Set #6

Due: Thursday 06/28/12

Since we haven't shown everything what is covered in the book, do not just cite random theorems from there. Use only what we have done in the lecture and prove or derive all other results you use. Once more, you will not get full points for calculations where you write only numbers. Make sure your whole solution is understandable.

Problem 1 *Volume calculations*

(a) W&A p. 448 no. 9: A segment of a cylindrical metal pipe is 120in long, has inside radius 5in and is 0.1in thick. Find the volume of metal in the segment.

(b) W&A p. 453 no. 6: Find the volume of a spherical shell with inner radius r and other radius $r+h$. Show that if h is small in comparison with r, the volume of the shell is approximately $4\pi r^2 h$.

(c) Two cylinders or two cones are called similar if the ratio of their radii is equal to the ratio of their altitudes, see p. 457. Do W&A p. 458 no. 3: A cone has height 10 and radius 6. A plane parallel to the base meets the altitude 3 above the base. Find the radius of the circle of intersection of the plane and the surface of the cone. Calculate the ratio of the volumes of the original cone and the cone with base the new circle.

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Problem 2 *Coordinate calculations*

(a) W&A p. 303 no. 3: The vertices of a quadrilateral are $A = (0,0), B = (x_1, 0), C = (x_2, y_2)$ and $D = (x_3, y_3)$. Show that if the diagonals AC, BD have the same midpoint, then $y_3 = y_2$ and $x_3 = x_2 - x_1$.

(b) W&A p. 310 no. 6: A, B and C have coordinates $(2,3), (4,7)$ and $(-6,7)$, respectively. Show by calculating the slopes of the sides that $\angle A$ is a right angle. Verify that if M is the midpoint of BC, then $AM = \frac{1}{3}BC$.

(c) W&A p. 327 no. 22: Find the equation of the line through $(-2,-1)$ perpendicular to $x + 4y = 0$.

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Problem 3 *Hyperbolas*

(a) Sketch the locus of points $\{(x, y) \in \mathbb{R}^2 | xy = c\}$ for $c < 0, c = 0$ and $c > 0$.

(b) Draw the three sketches from (a) into one diagram and indicate how the graph "moves" if you change c from $+\infty$ to $-\infty$.

(c) W&A p. 320 no. 13: Show the region of the coordinate plane containing points whose coordinates satisfy the conditions $1 \leq 8, y \geq 0$ and $y \leq \frac{4}{x}$.

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Problem 4 *Locus problems*

Do not prove your statements. However, draw an appropriate picture (not a construction but a sketch), from which it is possible to see what you have written. Take care that it is clear and try to imagine you draw this for one of your students.

(a) W&A p. 337 no. 10: What is the locus of points which are less than 3in from a fixed point O (i) in a plane containing O and (ii) in space?

(b) W&A p. 337 no. 16: What is the locus of points which are equidistant from two perpendicular planes?

(c) W&A p. 338 no. 13: What is the locus of the midpoints of line segments joining a given point P to points of a given line XY?

(d) W&A p. 347 no. 3: O is a fixed point on a plane X. What is the locus of points which are 5in from O and 3in from X?

(e) W&A p. 349 no. 5: Show how to locate in the plane of $\triangle ABC$ a point which is equidistant from A and B, and also equidistant from AB and AC.

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Problem 5 *Easy applications of trigonometry*

Make sure you prove or derive all the formulas we haven't had in the lecture. Make good sketches. This is not about scanning the book or the internet for the right formula and calculating, it is about how you solve those kind of problems with nothing but standard knowledge.

(a) W&A p. 412 no. 1: $\triangle ABC$ has $a = 12, b = 12$ and $c = 10$. The bisectors of the angles meet at I. Find the distance of I from AB.

(b) W&A p. 413 no. 9: AN is the altitude from A to BC in the triangle $\triangle ABC$, where the angles are called α, β and γ as usual. Let $AN = h$. Show that $a = h \cot \beta + h \cot \gamma$, where $\cot x := \frac{1}{\tan x}$.

(c) W&A p. 414 no. 9 plus a little bit extra: A rectangular box has base 30in by 15in, and heigth 20in. Find the angle which a diagonal of the box makes with the base. Additionally, find the length of the diagonal of the box.

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Problem 6 *Application of trigonometry in space*

W&A p. 413 no. 6b: OA, OB, and OC are three mutually perpendicular lines. $AO = 18, BO = 15$ and $CO = 20$. Find the angle between the planes ABC and BOC.

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Problem 7 *Area and perimeter of segments of circles*

Use only what we have shown in the lecture and derive every other formula you use.

(a) Suppose you are given a segment of a circle with radius r bounded by a part of the circle and two radii which enclose an angle α, measured in radians. Derive a formula for the area of the segment and the perimeter of the segment. Afterwards, express the corresponding formula for α in degrees.

(b) W&A p. 428 no. 8: $\triangle ABC$ is an equilateral triangle inscribed in a circle of radius 6. On AC as diameter a second circle is drawn. Find the area of the shaded crescent, see W&A.

(c) W&A p. 428 no. 9: $\square ABCD$ is a square of side 8, and P, Q, R, S are the midpoints of its sides. The arc DB has center A, and the arcs DO and OB have centers R and Q, respectively. Find the area of the region of the diagram that is shaded. See W&A for the picture.

(d) W&A p. 429 no. 12: The diagram shows two pulley wheels with a belt stretched tightly between them. Find the length of the belt if the radii of the circles are 4 and 20 and their centers are 32 apart. See W&A for the picture.

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Problem 8 *Volume of a cone*

W&A p. 413 no. 6b: A rectangular solid has dimensions $8, 6$ and 3. It rests with one of its largest faces on a horizontal plane. The base of a right circular cone is on the same horizontal plane, and the surface of the cone contains the four upper vertices of the rectangular solid. If the radius of the cone is r, express the volume of the cone in terms of r. *Hint: Make a good drawing. Figure out which parameters of the cone you need and try to find them using planar geometry in the appropriate planes.*

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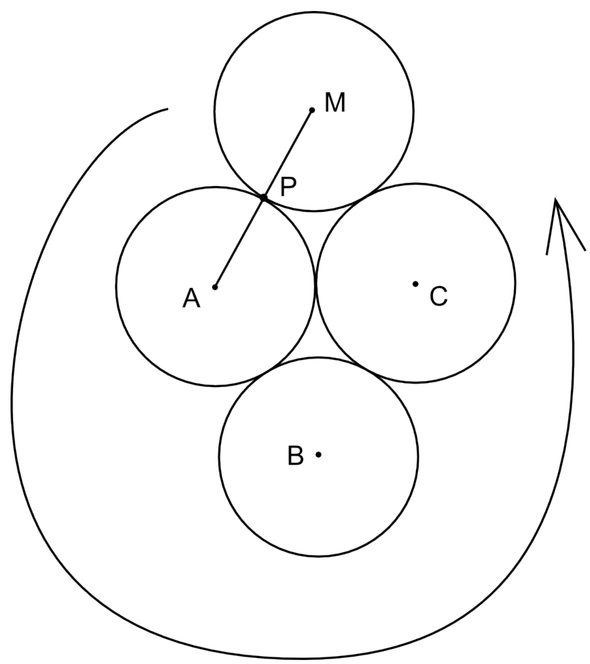
Problem 9 *(Bonus) LMO 380736*

In the picture you see three congruent circles with centers A, B and C, whcih remain fixed for the whole problem. A fourth congruent circle with center M is touching the two circles on top at the beginning. Imagine the circles as coins. The coin with center M is then moved around the figure consiting of the other three circles such that it touches the circles all the time and such that it rolls but does not glide. Call the contact point of the circles M and A at the beginning P. How often does the vector $\vec{MP}$ turn around itself if the coin gets moved around the other three once?



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