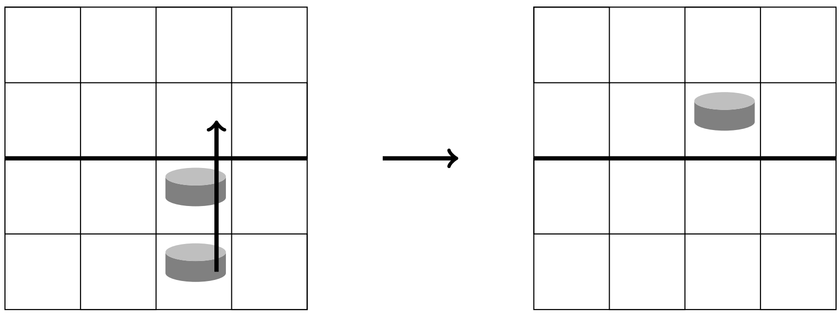
# Conway's Soldiers

In these notes we want to analyze a game called "Conway's Soldiers". They were written by Sven Prüfer for the International Kangaroo Camp 2014 at Werbellinsee close to Berlin, Germany. The main source is Julian Havils *Verblüfft?! Mathematische Beweise unglaublicher Ideen*, Springer.

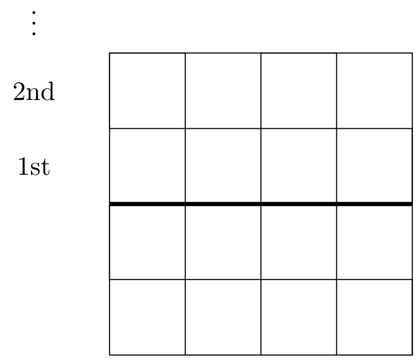
## The Rules of Conway's Soldiers

You are given an *infinite rectangular checkerboard* which is divided into two parts by a *horizontal line* called the base line. On each square of the grid there can be a game piece (called a *"soldier"*) or not. At the beginning there is an arbitrary amount of cells below the base line occupied by soldiers (possibly all cells). The game is played by a single person. In each turn only one move is allowed:

1. A soldier can jump over an adjacent soldier horizontally or vertically into an empty cell. The jumped-over soldier has to be removed.



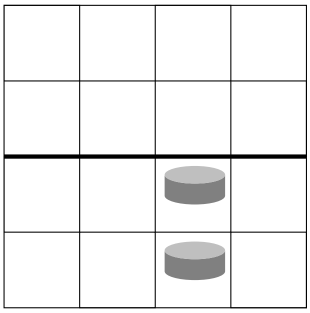
The initial condition, i.e. which cells below the base line are occupied by soldiers at the beginning can be chosen in any way. The goal is to move soldiers as far up as possible by using the rule above. We will denote rows as follows:




The goal of the game is to bring a soldier to the highest row possible and to determine the minimal number of soldiers at the beginning necessary to achieve this row.

## First Steps

Let us begin with the first row. This is easy as we need only two soldiers:



**Problem 1.** Try to find the minimal numbers of necessary soldiers in order to reach rows two, three and four.

| **Row** $n$ | **Necessary number of soldiers** |
|---|---|
| 2 | 2 |
| 3 | |
| 4 | |

Now the question is how does this continue. Try reaching row five.

## Convergent Power Series

It turns out that it is *impossible* to reach row five. We will prove this in the following chapter. However, in order to understand the proof we will need some background on convergent power series. Here we will not prove general statements and instead only look at examples in order to get a feeling for those objects.

A *polynomial* is a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ with $f(x) = \sum_{i=0}^{n} a_i x^i$ where $\sum_{i=0}^{n} b_i := b_0 + b_1 + b_2 + b_3 + \cdots + b_n$ and $n$ is called the *degree* of the polynomial if $a_n \neq 0$. For example, a linear polynomial is a polynomial of degree 1, i.e. it is given by $f(x) = a_1 x + a_0$.

Now imagine that our sum goes to infinity, i.e. $f(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots$. Ignoring the precise definition you can imagine this as follows. Given an $x \in \mathbb{R}$ the value $f(x)$ is the limit of the sequence $\sum_{i=0}^{N} a_i x^i$ for $N$ going to infinity. As usual with limits of sequences this may exist or not. For example if $a_i = 1$ for all $i \in \mathbb{N} \cup \{0\}$ and $x = 1$ we have $f(x) = \sum_{i=0}^{\infty} 1 = 1 + 1 + 1 \cdots$ which is certainly not finite.

*Example* (Decimal expression). Whenever you write real numbers (here $0 \leq x \leq 10$) as decimal expressions then you are in fact writing them as a convergent power series

$$\begin{aligned} x &= a_0 \cdot \left(\frac{1}{10}\right)^0 + a_1 \cdot \left(\frac{1}{10}\right)^1 + a_2 \cdot \left(\frac{1}{10}\right)^2 + \cdots \\ &= \sum_{i=0}^{\infty} a_i 10^{-i}, \end{aligned}$$

where the $a_i \in \{0, \ldots, 9\}$ are the $i$th digit behind the decimal point.

*Example* (Geometric Series). It can be shown that $\sum_{i=0}^{\infty} x^i = 1 + x + x^2 + x^3 + \cdots$ is finite whenever $|x| < 1$. In particular we can calculate its value. We will first apply the trick in the finite case.

Suppose $S = 1 + x + x^2 + x^3 + \cdots + x^{n-1}$. Then we can multiply this by $x$ and write it as follows

$$\begin{aligned} S &= 1 + x + x^2 + x^3 + \cdots + x^{n-1} \\ xS &= \phantom{1 + {}} x + x^2 + x^3 + \cdots + x^{n-1} + x^n. \end{aligned}$$

But then we have $S - xS = 1 - x^n$ and thus

$$\sum_{i=0}^{n-1} x^i = 1 + x + x^2 + x^3 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x}.$$

**Problem 2.** Use the same trick as in the example above to conclude the geometric sum formula

$$\sum_{i=0}^{\infty} x^i = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1 - x},$$

whenever $|x| < 1$.

**Solution.**

## The Golden Ratio

**Definition.** Define $y$ as the positive solution of the equation $y^2 + y - 1 = 0$.

*Remark.* This $y$ is actually equal to the inverse of the golden ratio $\varphi$ which is the solution of $\varphi^2 - \varphi - 1 = 0$. Thus we have $y \sim 0.61803$ whereas $\varphi \sim 1.61803$.

**Problem 3.** Show that $y^2 + y^3 + y^4 + \cdots = 1$.

**Solution.**

## Proof for Five Rows

The goal of these notes is to prove

**Theorem.** *In Conway's soldiers it is impossible for a soldier to reach row five in a finite amount of rounds.*

*Proof.* Remember that in order to reach row five it is enough to have one soldier on that row. Fix a cell on row five and call it the *target cell.* We will measure the distance on the board from a given cell to the target cell by the so-called *taxi metric.* Each cell has as a distance a power of $x$ to the order of the sum of the differences of rows and columns between the cell and the target cell. In the following picture, $x^0$ is the target cell because it is zero rows and zero columns far from itself.

| | | | | | |
|---|---|---|---|---|---|
| $x^2$ | $x^1$ | $x^0$ | $x^1$ | $x^2$ | $x^3$ |
| $x^3$ | $x^2$ | $x^1$ | $x^2$ | $x^3$ | $x^4$ |
| $x^4$ | $x^3$ | $x^2$ | $x^3$ | $x^4$ | $x^5$ |
| $x^5$ | $x^4$ | $x^3$ | $x^4$ | $x^5$ | $x^6$ |
| $x^6$ | $x^5$ | $x^4$ | $x^5$ | $x^6$ | $x^7$ |
| $x^7$ | $x^6$ | $x^5$ | $x^6$ | $x^7$ | $x^8$ |

Given soldiers on certain cells we denote the value of the game position by the sum of those monomials (i.e. those powers of $x$ as explained).

**Problem 4.** Convince yourself that any move in Conway's soldiers does one of the following for a suitable $n \in \mathbb{N} \cup \{0\}$ in the sum of the position:

1. $x^{n+2} + x^{n+1}$ is replaced by $x^n$
2. $x^n + x^{n-1}$ is replaced by $x^n$
3. $x^n + x^{n+1}$ is replaced by $x^{n+2}$

If we choose $x = y$ as above, i.e. as the positive solution of $x^2 + x - 1 = 0$ we can show the following

**Problem 5.** Show that for each possible move in Conway's soldiers the total sum of the position eiter stays the same or becomes smaller for our chosen $x$.

**Solution.**

Now we calculate the sum of various game positions.

**Problem 6.** Calculate the sum of the monomials for the completely occupied row below the target cell.

**Solution.**

**Problem 7.** Calculate the total sum of the monomials for a completely occupied lower half of the checkerboard below the target cell.

**Solution.**

**Problem 8.** Calculate the total sum of the monomials for a completely occupied lower half of the checkerboard five rows below the target cell. This is in fact the case of a fully covered half checkerboard with the target cell in the fifth row.

**Solution.**

We see that for our value of $x$ this sum is in fact equal to 1. Now the value of a soldier in the target cell is $x^0 = 1$. This means that if a single soldier is missing the total value of the initial position is less than 1. Since we have shown that in each move the value of the position decreases or stays constant this can never increase to one. Therefore it is impossible to reach the fifth row with finitely many soldiers and in particular in a finite amount of rounds. □

*Remark.* In fact it is possible to show that if the complete lower half is occupied by soldiers then you can reach row five in some limit sense.